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Finitely Generated Ideals in $A^\infty(D)$ in Pseudoconvex Domains

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Abstract

Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$-boundary and $V$ be an analytic submanifold of a neighborhood of $D$ which meets $\partial D$ transversally. We prove the following: If $g_1, \cdots, g_N$ generate the $O$-ideal of $V$, then they generate $I_v$ over $A^\infty(D)$.

1. Introduction. Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$-boundary. Let $V$ be an analytic subvariety of a neighborhood of $\overline{D}$. If $\Gamma$ is the ideal of $V$, from the general theory of Oka-Cartan-Serre, it follows that if $h_1, \cdots, h_N \in \Gamma(D, F_v)$ generate $\Gamma$, at every point $p \in D$, then they generate $\Gamma$ over $\Gamma(D, O) = O(D)$. Let $D$ have a strictly pseudoconvex boundary and let $V$ be smooth near $\partial D$ and meet $\partial D$ transversally. Then P. de Bartolomeis and G. Tomassini proved that if $g_1, \cdots, g_N \in O(\overline{D})$ generate $\Gamma$, then $1 \in \Gamma(D, F_v) \cap A^\infty(D)$ is generated by $g_1, \cdots, g_N$ over $A^\infty(D)$, where $A^\infty(D) = O(D) \cap C^\infty(\overline{D})$. In this paper we extend their result to pseudoconvex domains.

2. In this section we cite the definitions and the lemmas from E. Amar[1].

DEFINITION 1. Let $\Omega$ be a $C^\infty$ submanifold of $\overline{D}$. We denote by $\Omega$ the sheaf of germs of holomorphic functions in $D$ which extend $C^\infty$ smoothly to $\overline{D}$.

DEFINITION 2. Let $U = \{U_i : i \in I\}$ be an open covering of $\overline{D}$. We say that $U$ is admissible if $\partial (U_i \cap D)$ is $C^\infty$ and if each $\overline{U_i}$ possesses in $\overline{D}$ a basis $\{U_i^\alpha\}$ for the neighborhood system such that $U_i^\alpha$ is pseudoconvex and $\partial (U_i^\alpha \cap D)$ is $C^\infty$.

DEFINITION 3. Let $\Omega$ be an analytic sheaf in $D$. We say that $\Omega$ is globally

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coherent if $\Omega$ is generated by a finite number of global functions $u = (u_1, \ldots, u_k)$ in $A^\infty(D)$ over $\mathbb{Z}$ and the sheaf of relations of $u$, $R(u)$ is generated by $\xi = (\xi_1, \ldots, \xi_k)$, the sheaf of relations of $\xi$, $R(\xi)$ is generated by $\xi_{i+1}$ by recurrence, for points near $\partial D$. E. Amar [1] proved the following lemmas.

**Lemma 1.** Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$-boundary. Then there exists an arbitrarily fine admissible covering.

**Lemma 2.** Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$-boundary. Let $D$ be an admissible covering of $D$ and $\Omega$ be a globally coherent sheaf in $D$. Then $H^p(U, \Omega) = 0$, $p \geq 1$.

3. Let $V'_1, V'_2$ be analytic subvarieties of a neighborhood $D'$ of $\overline{D}$ such that if we set $V'_j = V'_1 \cap V'_j$ we have for $j = 1, 2, 3$;
   (i) $\text{Sing } V'_j \cap \partial D = \emptyset$
   (ii) $V'_j$ intersects $\partial D$ transversally.
Assume also that $V'_1$ and $V'_2$ intersect transversally along $\partial D$ and let $X' = V'_1 \cup V'_2$, $X = X' \cap D$, $V_j = V'_j \cap D$, $j = 1, 2, 3$.
Then we have

**Proposition 1.** The sheaf $F^\infty_x$ of functions of $A^\infty(D)$ which are zero on $X$ is globally coherent.

**Proof.** Let $\xi \in X \cap \partial D$ and let $U$ be an admissible neighborhood of $\xi$ on which we can choose complex coordinates $z_1, \ldots, z_n$ in such a way that

$V_1 \cap U = \{ z_1 = \cdots = z_k = 0 \}$
$V_s \cap U = \{ z_s = \cdots = z_m = 0 \}, s \leq k + 1$.

Because of the transversality, this is possible (See P. de Bartolomeis and G. Tomassini [3]). Let $f \in A^\infty(U \cap D)$ and $f|_X \cap U = 0$. Then we have

$f(z) = \sum_{j=1}^{k} z_j f_j(z), f_j \in A^\infty(U \cap D)$ and

$f(z) = \sum_{j=s}^{m} z_j g_j(z), g_j \in A^\infty(U \cap D)$. 
Therefore we can write \( f(z) = \sum_{j=1}^{m} z_{j} h_{j}(z) \), \( h_{j} \in \Lambda_{\infty}(U \cap D) \). In order to prove that \( F_{\infty} \) is globally coherent, we have to show that the complex

\[
0 \to \Lambda^{n}(Z^{n}) \xrightarrow{\phi} \Lambda^{n-1}(Z^{n}) \xrightarrow{\phi} \cdots \xrightarrow{\phi} \Lambda^{1}(Z^{n}) \xrightarrow{\phi} \Lambda^{0}(Z^{n}) \xrightarrow{R_{X}} \Lambda^{1}(Z^{n}) \mid_{X \cap D} \text{ is exact.}
\]

We follow the proof of E. Amar [1]. Let \( \{e_{1}, \ldots, e_{m}\} \) be a basis of \( \Lambda^{1}(Z^{n}) \). If \( h \in \Lambda^{1}(Z^{n}) \), then \( h = \sum_{i=1}^{m} h_{i} e_{i} \), \( h_{i} \in Z \). We define \( \phi(e_{i}) = z_{i} \), \( i = 1, \ldots, m \), \( \phi(e_{i} \Lambda e_{j}) = z_{i} e_{a} - \phi(e_{a}) \Lambda e_{j} \), where \( e_{a} = e_{a_{1}} \Lambda \cdots \Lambda e_{a_{l}} \) with \( a_{i} \leq j \). We have to show that (2) If \( f \in \Lambda^{l-1} \), then there exists \( h \in \Lambda^{j} \) such that \( f = \phi(h) \).

Suppose (2) is true for \((m-1, n-1)\). Let \( f \in \Lambda^{l-1}(Z^{n}) \) with \( \phi(f) = 0 \). We have

\[
f = \sum_{|\alpha| = j-1} a_{\alpha} e_{\alpha} + \sum_{|\beta| = j-2} b_{\beta} e_{\beta} \Lambda e_{m}.
\]

Let \( \tau \) be a mapping of \( \Lambda^{l-1}(Z^{n}) \) in \( \Lambda^{l-1}(\tilde{Z}^{n-1}) \) defined by \( \tau(f) = \sum_{|\alpha| = j-1} a_{\alpha}(z_{1}, \ldots, z_{m-1}, 0, z_{m+1}, \ldots, z_{n})e_{\alpha} \), where \( \tilde{Z} \) is the sheaf of germs of holomorphic functions in \( D \setminus \{z_{m} = 0\} \) which are \( C^{\infty} \) up to the boundary. Then \( \phi(\tau f) = 0 \). By the induction hypothesis, there exists \( \tilde{h} \in \Lambda^{j}(\tilde{Z}^{n-1}) \) such that \( \tau(f) = \phi(\tilde{h}) \) with \( \tilde{h} = \sum_{|\alpha| = j} \tilde{h}_{\alpha} e_{\alpha} \). We can extend \( \tilde{h}_{\alpha} \) to \( h_{\alpha} \) in \( Z \), and we set \( h = \sum_{|\alpha| = j} h_{\alpha} e_{\alpha} \in \Lambda^{j}(Z^{n}) \).

Then \( \phi(h) = \sum_{|\alpha| = j} c_{\alpha} e_{\alpha} \). From this we have \( \tau(f - \phi(h)) = 0 \). Then we have

\[
f - \phi(h) = \sum_{|\alpha| = j-1} a_{\alpha} e_{\alpha} + \sum_{|\beta| = j-2} b_{\beta} e_{\beta} \Lambda e_{m}.
\]

From this we can write \( a_{\alpha} = c_{\alpha} - z_{m} d_{\alpha} \) with \( d_{\alpha} \in Z \). Thus we have

\[
f - \phi(h) = \sum_{|\alpha| = j-1} z_{m} d_{\alpha} e_{\alpha} + \sum_{|\beta| = j-2} b_{\beta} e_{\beta} \Lambda e_{m}.
\]

Therefore

\[
0 = \phi(f) = \sum_{|\alpha| = j-1} z_{m} d_{\alpha} \phi(e_{\alpha}) + \sum_{|\beta| = j-2} b_{\beta} z_{m} e_{\beta} - \sum_{|\beta| = j-2} b_{\beta} \phi(e_{\beta}) \Lambda e_{m}.
\]

Hence \( \sum_{|\alpha| = j-1} d_{\alpha} \phi(e_{\alpha}) + \sum_{|\beta| = j-2} b_{\beta} e_{\beta} = 0 \). We set

\[
q = \sum_{|\alpha| = j-1} d_{\alpha} e_{\alpha} \Lambda e_{m} \in \Lambda^{j}(Z^{n}).
\]

Then we have

\[
\phi(q) = \sum_{|\alpha| = j-1} d_{\alpha} z_{m} e_{\alpha} - \sum_{|\alpha| = j-1} d_{\alpha} \phi(e_{\alpha}) \Lambda e_{m} = f - \phi(h).
\]

Thus \( f = \phi(h + q) \). If \( \xi \in \partial D \mid X \), then one of the \( z_{i} \)'s is not zero. Suppose
Let $f \in \Lambda^{l-1}(Z^n)$, $\phi(f) = 0$. Then
\[
f = |\sum_{m \in \alpha} a_m e_\alpha + i \sum_{j=1}^{\beta} b_\beta e_\beta|\Lambda e_m.\]
We set $d_\alpha = -\frac{a_\alpha}{z_m}$ and
\[
q = |\sum_{m \in \alpha} d_\alpha e_\alpha|\Lambda e_m \in \Lambda^l(Z^n). \quad \text{Then } f = \phi(q). \quad \text{Therefore } F_x^\infty \text{ is globally coherent.}
\]

**PROPOSITION 2.** *The restriction homomorphism*
\[A^\infty(D) \to A^\infty(X)\]

*is onto.*

**PROOF.** Let $f \in A^\infty(X)$. First of all we show that if $\zeta \in \partial D \cap X$, then there exists an admissible neighborhood $U_\zeta$ of $\zeta$ in $\overline{D}$ such that $f|_{U_\zeta \cap X}$ admits an extension $F_\zeta$ in $A^\infty(U_\zeta \cap D)$. We can choose complex coordinates $z_1, \ldots, z_n$ on $U_\zeta$ in such a way that
\[
V_i \cap U_\zeta = \{z_1 = \cdots = z_k = 0\}
\]
\[
V_s \cap U_\zeta = \{z_s = \cdots = z_n = 0\}, s \leq k + 1.
\]
Let $f_i = f|_{V_i \cap U_\zeta}$. By applying the extension theorem obtained by E. Amar [1], we can write
\[
f_i = \sum_{j=k+1}^{m} g_{ij} z_j, \quad g_{ij} \in A^\infty(D \cap U_\zeta),
\]
\[
f_i = \sum_{j=1}^{s} h_{ij} z_j, \quad h_{ij} \in A^\infty(D \cap U_\zeta).
\]
We set $F = \sum_{j=1}^{m} h_{ij} z_j + \sum_{j=k+1}^{m} g_{ij} z_j$. Then $F \in A^\infty(D \cap U_\zeta)$ and $F|_{X \cap U_\zeta} = f$. If $\zeta \in \partial D \cap X$, there exists an admissible neighborhood $U_\zeta$ of $\zeta$ such that $\overline{U_\zeta} \cap X = \phi$. In this neighborhood we take $F_\zeta = 0$. Since $\overline{D}$ is compact, we have finitely many $U_\zeta$ which cover $\overline{D}$. We denote by $U = \{U_i : i \in I\}$ this covering and by $F_i$ the corresponding functions. Then $G_{ij} = F_i - F_j$ is a 1-cocycle with values in $F_\zeta$. Since $H^1(U, F_\zeta) = 0$, there exists a 0-cochain $(H_i : i \in I)$ in $F_\zeta^\infty$ such that $G_{ij} = (\partial H_i)_j$. Then $\tilde{f} = F_i - H_i$ on $U_i$ is globally defined and $\tilde{f}|_X = f$, $f \in A^\infty(D)$. Therefore proposition 2 is proved.

**COROLLARY.** *Let $0 \in D$ and assume the coordinate space $L = \{z \in \mathbb{C}^n : z_{s+1} = \cdots = z_n = 0\}$ intersects $\partial D$ transversally. Then every $f \in A^\infty(D)$ vanishing on $L \cap D$ can be written as $f = \sum_{j=k+1}^{n} h_{j} z_j, h_{j} \in A^\infty(D).$* 

**PROOF.** In the case when $k = n$, we set $\tilde{f}(z_1, \ldots, z_n) = f(z_1, \ldots, z_n) z_n^{-1}.$
Then $\vec{f}(z_1, \ldots, z_n) \in A^\infty(D)$. Assume our corollary is proved when codimension of $L$ is $\leq k$, and let $D_i = D \cap \{ z_i = 0 \}$. By the induction hypothesis we have

$$f|_{D_i} = \sum_{j = k+1}^{n-1} \lambda_j z_j, \lambda_j \in A^\infty(D_i).$$

Then there exist $A_j \in A^\infty(D)$ such that $A_j \mid D_i = \lambda_j$. Then $f - \sum_{j = k+1}^{n-1} A_j z_j$ vanishes on $D_i$ and so there is $A_\infty \in A^\infty(D)$ such that $f - \sum_{j = k+1}^{n-1} A_j z_j = A_\infty z_n$. Therefore our corollary is proved.

**THEOREM.** Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$-boundary and let $D'$ be an open neighborhood of $D$ and $V'$ an analytic subvariety of $D'$, with $V = V' \cap D$. Let $\{ g_1, \ldots, g_k \}$ be a complete system of defining functions for $V'$. Assume that $\text{Sing } V' \cap \partial D = \emptyset$ and $V'$ intersects $\partial D$ transversally. Then every $f \in A^\infty(D)$ vanishing on $V$ can be written as $f = \sum_{j=1}^k h_j g_j$, where $h_1, \ldots, h_k \in A^\infty(D)$.

**PROOF.** We follow the proof of P. de Bartolomeis and G. Tomassini ([3], theorem 3.1). Consider the map $g : D' \to \mathbb{C}^k$ given by $g(z) = (g_1(z), \ldots, g_k(z))$ and let $\Gamma_* = \{(z,w) \in D' \times \mathbb{C}^k : w_j - g_j(z) = 0, 1 \leq j \leq k\}$ be its graph. Consider in $D' \times \mathbb{C}^k$ a pseudoconvex bounded domain $B = \{(z,w) \in C' \times C^k : \text{r}(z) + \exp \left( \sum_{j=1}^k w_j - w_j \right) < 0\}$, where $r(z)$ is the defining function of $D$. Then we have

(i) $B \cap (D' \times \{0\}) = D$

(ii) $\Gamma_* \cap \partial B$ transversally.

Let $X = (D' \cap \Gamma_*) \cap B$ and let $f \in A^\infty(D)$ be such that $f|_V = 0$. We define $F \in A^\infty(X)$ by $F = f$ on $D$, $F = 0$ on $\Gamma_* \cap B$. By proposition 2, we can find $G \in A^\infty(B)$ such that $G|_X = F$; in particular $G|_{\Gamma_* \cap B} = 0$. Now $\Gamma_* \cap B$ is holomorphically equivalent to a plane section and thus, using corollary, we can find $\tilde{h}_1, \ldots, \tilde{h}_k \in A^\infty(B)$ such that $G = \sum_{j=1}^k \tilde{h}_j (g_j - w_j)$. Therefore if we set $h_j = \tilde{h}_j|D$ we get $G|D = f = \sum_{j=1}^k h_j g_j$. Therefore our theorem is proved.

References
