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Topological Types of Paracompact Connected 1-manifolds

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Abstract

A paracompact connected 1-manifold is homeomorphic to either an interval on the real line or a circle.

In this note we give a proof of the following

THEOREM. Any paracompact connected 1-manifold with or without boundary is homeomorphic to an interval on the real line or a circle.

By the hypothesis "paracompactness" non-standard spaces such as non-Hausdorff connected 1-manifold ([4], p. 225) and the so called "long line" ([4], p. 159) —— it is not paracompact —— are excluded. In the paper [2] we have used the theorem without proof to determine topological types of homeomorphism groups on paracompact connected 1-manifold. Here we give a proof of it in order to make sure. The proof is carried out in expectation that any paracompact connected 1-manifold will be obtained by joining at most countable number of (open, closed, or half-open) arcs without making branches.

Proof of Theorem. Let $M$ be a paracompact connected 1-manifold with or without boundary, $R$ the real line, $R$, a half-open interval on $R$, $I$ a closed unit interval on $R$, and $S$ a circle.

Step 1°. $M$ is metrizable by Smirnov’s metrization theorem ([4], p. 260). As a connected locally compact metrizable space, $M$ is separable ([3], Appendix 2). Therefore $M$ is second-countable. As a locally compact Hausdorff space with a countable basis having the topological dimension 1, $M$ can be imbedded as a closed subset of the euclidean 3-space ([4], p. 315) —— we consider $M$ such a subspace hereafter.

Since $M$ is second-countable 1-manifold, there exists a countable covering of $M$, consisting of open neighborhoods $(U_i)$ and $(W_k)$ in $M$ as follows: i) $U_i=\mathbb{R}$, CI$U_i=I$, and just one of the two end points of CI$U_i$ is a boundary point of $M$. 
Let $x_i$ be the boundary point of $M$ in $U_i$, then $x_i \sim x_j$ if $i \sim j$. ii) $W_k \sim R$, $\text{Cl} W_k \sim I$, and both of the two end points of $\text{Cl} W_k$ are not boundary points of $M$. Each $W_k$ is not contained in any $U_i$. Here "\sim" means "homeomorphic to", and there are possibly no $U$'s or no $W$'s.

Step 2° (Join of two $U$'s). If $U_i$ intersects $U_j$ for some different $i$ and $j$, then $M = U_i \cup U_j \sim I$.

Proof. For convenience let $U_i = U$, $U_j = V$, $x_i = x$, $x_j = y$, and let $a$ and $b$ be the end point of $\text{Cl} U_i$ and $\text{Cl} U_j$ respectively which is not a boundary point of $M$. Take a point $p$ of $U \cap V$. There exist open neighborhoods $O_\alpha$ of $p$ in $M$ such that $O_\alpha \subset U \cap V$ and $O_\alpha \sim R$. Let $O^*$ be the union of all such $O_\alpha$, then $O^*$ is the maximum of open neighborhoods of $p$ in $M$ which are contained in $U \cap V$ and homeomorphic to $R$. In fact $p \in O^* \subset (\text{open arc } xa) \sim R$ and $O^*$ is open and connected, and so $O^*$ is homeomorphic to an open interval on $R$. We show that $\text{Cl} O^* \sim I$ and end points of $\text{Cl} O^*$ are $\{a, b\}$. Noting that $\text{Cl} O^* \subset (\text{closed arc } xa) \sim I$, we see $\text{Cl} O^*$ is homeomorphic to a closed interval of $I$. Let $q$ and $r$ be the end points of $\text{Cl} O^*$. Each of $q$ and $r$ differs from $x$ and $y$. Changing notations $q$, $r$ if necessary, let the orientations of $\overrightarrow{qr}$ and $\overrightarrow{yb}$ on the arc $yb$ coincide. We can show that the orientations of $\overrightarrow{qr}$ and $\overrightarrow{ax}$ are the same on the arc $xa$. If not, we would arrive at a contradiction using the maximum property of $O^*$ and the fact that $q$ is an inner point of arc $xp$, arc $yp$, and 1-manifold $M$ respectively. Moreover it follows that $q = a$ and $r = b$. In the result, $U \cup V$ is the union of three arcs $xb$, $ba$, and $ay$ only adjacent arcs of which have a common end point. Noting that $M$ is connected and the property of $U$ and $V$, we have $M = U \cup V$.

Step 3° (Join of $U_i$ and $W_k$). If $U_i$ intersects $W_k$, then $U_i \cup W_k \sim R$, $\text{Cl} (U_i \cup W_k) \sim I$, and just one of the end points of $\text{Cl} (U_i \cup W_k)$ is a boundary point of $M$.

Proof. The similar proof as in step 2° is valid.

Step 4° (Join of two $W$'s). If $W_k$ intersects $W_l$, then exactly one of the following conclusions holds: i) $W_k \cup W_l \sim R$, $\text{Cl} (W_k \cup W_l) \sim I$, and no end points of $\text{Cl} (W_k \cup W_l)$ are boundary points of $M$. ii) $W_k \cup W_l \sim S$. There are no $U$'s and no another $W$'s, and $M \sim S$.

Proof. Let $c_k$ and $d_k$ (resp. $c_l$ and $d_l$) be two end points of $\text{Cl} W_k$ (resp. $\text{Cl} W_l$). For any fixed point $p$ of $W_k \cap W_l$, there exists the maximum $O^*$ among all open neighborhoods of $p$ in $M$ which are contained in $W_k \cap W_l$ and homeomorphic to $R$. Then $\text{Cl} O^* \sim I$. Let $q$ and $r$ be the end points of $\text{Cl} O^*$. We can suppose that both of the orientations $\overrightarrow{c_k d_k}$ and $\overrightarrow{c_l d_l}$ coincide with that of $\overrightarrow{qr}$ on the arc $qr$. Then $q = c_k$ or $c_l$, and $r = d_k$ or $d_l$. In the case where $O^* \sim W_k \cap W_l$, the conclusion i) follows. If $O^* \sim W_k \cap W_l$, then there exists another set $O^{**}$ in $W_k \cap W_l$ which has the similar properties as the above $O^*$. In this case $W_k \cap W_l = O^* \cup O^{**}$, and $M = W_k \cup W_l \sim S$. 
Step 5 (Join of a countably infinite number of $W$'s, whose union is not homeomorphic to a circle). Let $\{W'_k \mid k=1,2,\ldots\}$ be an expanding sequence of open neighborhoods in $M$ such that $W'_k \supseteq R$, $\text{Cl}(W'_k) = I$, and no end points of $\text{Cl}W'_k$ are boundary points of $M$, then $\bigcup_{k=1}^{\infty} W'_k = R$.

Proof. Let $X = \bigcup_{k=1}^{\infty} W'_k$. We can naturally define a simple order on $X$. $X$ is non-degenerate and has no smallest and no largest element. Give $X$ the induced order topology, and it coincides with the subspace topology. Then $X$ is second-countable, locally compact, connected, and merizable. Let $a$ and $b$ be any two fixed points outside $X$, and let $X^* = X \cup \{a, b\}$. Define a simple order on $X^*$ such that it is the same on $X$ and $a < x < b$ for every $x \in X$. $a$ (resp. $b$) is the smallest (resp. largest) element of $X^*$. Give $X^*$ the induced order topology, then $X^*$ is connected and only two points $a, b$ are the non-cut points of $X^*$. We can see that $X^*$ is compact as follows. Let $\{V_{\lambda} \mid \lambda \in \Lambda\}$ be any open covering of $X^*$. For some $\mu, \nu \in \Lambda$, $a \in V_{\mu}$ and $b \in V_{\nu}$. Take $x, y \in X$ such that $[a, x] \subseteq V_{\mu}$, $(y, b] \subseteq V_{\nu}$, and $x < y$. $[x, y]$ is connected. Since $X$ is separable and locally euclidean, there exist a countable dense subset $\{p_n\}$ of $X$ and, for each $p_n$ an open interval $O(n)$ containing $p_n$ whose closure in $X$ is compact. By connectedness of $[x, y]$, there exists a finite subcollection of $\{O(n) \cap [x, y] \mid n=1, 2, \ldots\}$ which is a simple chain from $x$ to $y$, say $O(n_1) \cap [x, y], \ldots, O(n_k) \cap [x, y]$. We can prove that these sets covers $[x, y]$. Thus $[x, y] \subseteq (\text{Cl}O(n_1) \cup \cdots \cup \text{Cl}O(n_k))$, and so $[x, y]$ is compact. Therefore we can cover $[x, y]$ by a finite number of $V$'s. Hence $X^*$ is compact. Now we can see that $X^*$ is metrizable also. As a non-degenerate, compact, connected, metrizable space, $X^*$ is homeomorphic to $I$ ([1], p. 168). Since two end points $a, b$ of $X^*$ are the only non-cut points of $X^*$, $X = R$.

Step 6 (Settlement of our proof).

The case where $M$ is a 1-manifold without boundary—. Since $M$ is connected, rearranging $\{W_k\}$ if necessary, we can suppose that $W_1 \cup \cdots \cup W_k$ intersects $W_{k+1}$, for every $k$. Let $W'_k = W_1 \cup \cdots \cup W_k (k=1, 2, \ldots)$. Then by step 4, exactly one of the following conclusions holds:

i) For some $n$, $W_{n+1}, W_{n+2}, \ldots$ do not exist, and $M = W'_n$. In this case $M$ is homeomorphic to either $R$ or $S$.

ii) $\{W'_k \mid k=1,2,\ldots\}$ is an expanding infinite sequence of open neighborhoods as in step 5. And $M = R$.

The case where the boundary of $M$ consists of only one point—. If there are no $W$'s, then $M = U_1 = R$. If any $W$'s exists, we can suppose that $U_1 \cup W_1 \cup W_2 \cup \cdots \cup W_k$ intersects $W_{k+1}$, for every $k$. Let $U'_1 = U_1$, $U'_{k+1} = U_1 \cup W_1 \cup \cdots \cup W_k (k=1, 2, \ldots)$. By step 4 we see that $\bigcup_{k=1}^{m} W_k = S$ for some $m$, $\bigcup_{k=1}^{n} W_k = R$ for some $n$, or not.

The first case can not occur. And in the third case, by the similar proof as in
step 5° (adding one point to $\cup U_k$) we can see that $M = \cup U_k \approx R$.

The case where the boundary of $M$ contains at least two points. If any two $U_i$ and $U_j$ ($i \neq j$) intersect, then by step 2° $M = U_i \cup U_j \approx I$ and there is no other $U$'s and $W$'s. Now suppose that no two $U$'s intersect. Since $M$ is connected, $M - \cup U_i$ is nonempty and there exists at least one $W_k$. $U_1$ (resp. $U_2$) intersects $W_l$ (resp. $W_m$) for some $l$ (resp. $m$). Take points $z_1$ of $U_i \cap W_l$ and $z_2$ of $U_j \cap W_m$. There exists a finite subcollection of $\{U_i\} \cup \{W_k\}$ which is a simple chain from $z_1$ to $z_2$. There is no $U_j$ ($j \neq 1, 2$) between end links $U_i$ and $U_j$ in the chain, for if there was a $U_j$ between them, applying step 3° and step 2°, we would have $M \approx I$ without using $U_i$, this is a contradiction. Consequently applying step 3° and step 2° again to the chain we can see that $M \approx I$ and there is no $U$’s other than $U_i$ and $U_j$.

References