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Topological Types of Paracompact Connected 1-manifolds

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Abstract

A paracompact connected 1-manifold is homeomorphic to either an interval on the real line or a circle.

In this note we give a proof of the following

THEOREM. Any paracompact connected 1-manifold with or without boundary is homeomorphic to an interval on the real line or a circle.

By the hypothesis “paracompactness” non-standard spaces such as non-Hausdorff connected 1-manifold ([4], p. 225) and the so called “long line” ([4], p. 159) — it is not paracompact — are excluded. In the paper [2] we have used the theorem without proof to determine topological types of homeomorphism groups on paracompact connected 1-manifold. Here we give a proof of it in order to make sure. The proof is carried out in expectation that any paracompact connected 1-manifold will be obtained by joining at most countable number of (open, closed, or half-open) arcs without making branches.

Proof of Theorem. Let \( M \) be a paracompact connected 1-manifold with or without boundary, \( R \) the real line, \( R \), a half-open interval on \( R \), \( I \) a closed unit interval on \( R \), and \( S \) a circle.

Step 1°. \( M \) is metrizable by Smirnov’s metrization theorem ([4], p. 260). As a connected locally compact metrizable space, \( M \) is separable ([3], Appendix 2). Therefore \( M \) is second-countable. As a locally compact Hausdorff space with a countable basis having the topological dimension 1, \( M \) can be imbedded as a closed subset of the euclidean 3-space ([4], p. 315) — we consider \( M \) such a subspace hereafter.

Since \( M \) is second-countable 1-manifold, there exists a countable covering of \( M \), consisting of open neighborhoods \((U_i)\) and \((W_i)\) in \( M \) as follows: i) \( U_i=R \), \( \text{Cl}U_i=I \), and just one of the two end points of \( \text{Cl}U_i \) is a boundary point of \( M \).
Let $x_i$ be the boundary point of $M$ in $U_i$, then $x_i \sim x_j$ if $i \not\sim j$. ii) $W_k \approx R$, $\text{Cl} \ W_k \approx I$, and both of the two end points of $\text{Cl} \ W_k$ are not boundary points of $M$. Each $W_k$ is not contained in any $U_i$. Here "\approx" means "homeomorphic to", and there are possibly no $U$‘s or no $W$’s.

Step 2° (Join of two $U$’s). If $U_i$ intersects $U_j$ for some different $i$ and $j$, then $M = U_i \cup U_j \approx I$.

Proof. For convenience let $U_i = U, U_j = V, x_i = x, x_j = y$, and let $a$ and $b$ be the end point of $\text{Cl} \ U_i$ and $\text{Cl} \ U_j$ respectively which is not a boundary point of $M$. Take a point $p$ of $U \cap V$. There exist open neighborhoods $O_p$ of $p$ in $M$ such that $O_p \subset U \cap V$ and $O_p \approx R$. Let $O^*$ be the union of all such $O_p$, then $O^*$ is the maximum of open neighborhoods of $p$ in $M$ which are contained in $U \cap V$ and homeomorphic to $R$. In fact $p \in O^* \subset (\text{open arc } xa) \approx R$ and $O^*$ is open and connected, and so $O^*$ is homeomorphic to an open interval on $R$. We show that $\text{Cl} \ O^* \approx I$ and end points of $\text{Cl} \ O^*$ are $\{a,b\}$. Noting that $\text{Cl} \ O^* \subset (\text{closed arc } xa) \approx I$, we see $\text{Cl} \ O^*$ is homeomorphic to a closed interval of $I$. Let $q$ and $r$ be the end points of $\text{Cl} \ O^*$. Each of $q$ and $r$ differs from $x$ and $y$. Changing notations $q, r$ if necessary, let the orientations of $qr$ and $yb$ on the arc $yb$ coincide. We can show that the orientations of $qr$ and $ax$ are the same on the arc $xa$. If not, we would arrive at a contradiction using the maximum property of $O^*$ and the fact that $q$ is an inner point of arc $xp$, arc $yp$, and 1-manifold $M$ respectively. Moreover it follows that $q=a$ and $r=b$. In the result, $U \cup V$ is the union of three arcs $xb, ba,$ and $ay$ only adjacent arcs of which have a common end point. Noting that $M$ is connected and the property of $U$ and $V$, we have $M = U \cup V$.

Step 3° (Join of $U_i$ and $W_k$). If $U_i$ intersects $W_k$, then $U_i \cup W_k \approx R$, $\text{Cl} (U_i \cup W_k) \approx I$, and just one of the end points of $\text{Cl} (U_i \cup W_k)$ is a boundary point of $M$.

Proof. The similar proof as in step 2° is valid.

Step 4° (Join of two $W$’s). If $W_k$ intersects $W_l$, then exactly one of the following conclusions holds: i) $W_k \cup W_l \approx R$, $\text{Cl} (W_k \cup W_l) \approx I$, and no end points of $\text{Cl} (W_k \cup W_l)$ are boundary points of $M$ ii) $W_k \cup W_l \approx S$. There are no $U$’s and no another $W$’s, and $M \approx S$.

Proof. Let $c_k$ and $d_k$ (resp. $c_l$ and $d_l$) be two end points of $\text{Cl} \ W_k$ (resp. $\text{Cl} \ W_l$). For any fixed point $p$ of $W_k \cap W_l$, there exists the maximum $O^*$ among all open neighborhoods of $p$ in $M$ which are contained in $W_k \cap W_l$ and homeomorphic to $R$. Then $\text{Cl} \ O^* \approx I$. Let $p$ and $q$ be the end points of $\text{Cl} \ O^*$. We can suppose that both of the orientations $c_k d_k$ and $c_l d_l$ coincide with that of $qr$ on the arc $qr$. Then $q=c_k$ or $c_l$, and $r=d_k$ or $d_l$. In the case where $O^* = W_k \cap W_l$ the conclusion i) follows. If $O^* \approx W_k \cap W_l$, then there exists another set $O^{**}$ in $W_k \cap W_l$ which has the similar properties as the above $O^*$. In this case $W_k \cap W_l = O^* \cup O^{**}$, and $M = W_k \cup W_l \approx S$. 

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Step 5° (Join of a countably infinite number of W's, whose union is not homeomorphic to a circle). Let \( \{ W'_k | k = 1, 2, \ldots \} \) be an expanding sequence of open neighborhoods in \( M \) such that \( W'_k = R \), \( \text{Cl}(W'_k) = I \), and no end points of \( \text{Cl}(W'_k) \) are boundary points of \( M \), then \( \bigcup_{k=1}^{\infty} W'_k = R \). 

Proof. Let \( X = \bigcup_{k=1}^{\infty} W'_k \). We can naturally define a simple order on \( X \). \( X \) is non-degenerate and has no smallest and no largest element. Give \( X \) the induced order topology, and it coincides with the subspace topology. Then \( X \) is second-countable, locally compact, connected, and merizable. Let \( a \) and \( b \) be any two fixed points outside \( X \), and let \( X^* = X \cup \{a, b\} \). Define a simple order on \( X^* \) such that it is the same on \( X \) and \( a < x < b \) for every \( x \in X \). \( a \) (resp. \( b \)) is the smallest (resp. largest) element of \( X^* \). Give \( X^* \) the induced order topology, then \( X^* \) is connected and only two points \( a, b \) are the non-cut points of \( X^* \). We can see that \( X^* \) is compact as follows. Let \( \{V_\lambda | \lambda \in \Lambda \} \) be any open covering of \( X^* \). For some \( \mu, \nu \in \Lambda \), \( a \in V_\mu \) and \( b \in V_\nu \). Take \( x, y \in X \) such that \( [a, x] \subseteq V_\mu \), \( (y, b] \subseteq V_\nu \), and \( x < y \). \( [x, y] \) is connected. Since \( X \) is separable and locally euclidean, there exist a countable dense subset \( \{p_n\} \) of \( X \) and, for each \( p_n \) an open interval \( O(n) \) containing \( p_n \) whose closure in \( X \) is compact. By connectedness of \( [x, y] \), there exists a finite subcollection of \( \{O(n) \cap [x, y] | n = 1, 2, \ldots \} \) which is a simple chain from \( x \) to \( y \), say \( O(n_1) \cap [x, y], \ldots, O(n_m) \cap [x, y] \). We can prove that these sets covers \( [x, y] \). Thus \( [x, y] \subseteq (\text{Cl}(O(n_1)) \cup \ldots \cup (\text{Cl}(O(n_m)))) \), and so \( [x, y] \) is compact. Therefore we can cover \( [x, y] \) by a finite number of \( V \)'s. Hence \( X^* \) is compact. Now we can see that \( X^* \) is metrizable also. As a non-degenerate, compact, connected, metrizable space, \( X^* \) is homeomorphic to \( I \) ([1], p. 168). Since two end points \( a, b \) of \( X^* \) are the only non-cut points of \( X^* \), \( X = R \).

Step 6° (Settlement of our proof). 

The case where \( M \) is a 1-manifold without boundary. Since \( M \) is connected, rearranging \( \{ W_k \} \) if necessary, we can suppose that \( W_1 \cup \ldots \cup W_k \) intersects \( W_{k+1} \), for every \( k \). Let \( W'_k = W_1 \cup \ldots \cup W_k \) \( (k = 1, 2, \ldots) \). Then by step 4°, exactly one of the following conclusions holds:

i) For some \( n \), \( W_{n+1}, W_{n+2}, \ldots \) do not exist, and \( M = W'_n \). In this case \( M \) is homeomorphic to either \( R \) or \( S \).

ii) \( \{W'_{k'} | k' = 1, 2, \ldots \} \) is an expanding infinite sequence of open neighborhoods as in step 5°. And \( M = R \).

The case where the boundary of \( M \) consists of only one point. If there are no \( W \)'s, then \( M = U_1 = R \). If any \( W \) exists, we can suppose that \( U_1 \cup W_1 \cup W_2 \cup \ldots \cup W_k \) intersects \( W_{k+1} \), for every \( k \). Let \( U'_1 = U_1 \), \( U'_{k+1} = U_1 \cup W_1 \cup \ldots \cup W_k \) \( (k = 1, 2, \ldots) \). By step 4° we see that \( \bigcup_{k=1}^{m} W_k = S \) for some \( m \), \( \bigcup_{k=1}^{n} W_k = R \) for some \( n \), or not. The first case can not occur. And in the third case, by the similar proof as in
step 5° (adding one point to $\cup U_k$) we can see that $M = \cup U_k \approx R$.

The case where the boundary of $M$ contains at least two points ---. If any two $U_i$ and $U_j$ ($i \neq j$) intersect, then by step 2° $M = U_i \cup U_j \approx I$ and there is no other $U$’s and $W$’s. Now suppose that no two $U$’s intersect. Since $M$ is connected, $M - \cup U_i$ is nonempty and there exists at least one $W_k$. $U_l$ (resp. $U_r$) intersects $W_l$ (resp. $W_r$) for some $l$ (resp. $m$). Take points $z_1$ of $U_l \cap W_l$ and $z_2$ of $U_r \cap W_m$. There exists a finite subcollection of \{${U_i}$\} U \{${W_k}$\} which is a simple chain from $z_1$ to $z_2$. There is no $U_j$ ($j \neq 1, 2$) between end links $U_1$ and $U_2$ in the chain, for if there was a $U_j$ between them, applying step 3° and step 2°, we would have $M \approx I$ without using $U_1$, this is a contradiction. Consequently applying step 3° and step 2° again to the chain we can see that $M \approx I$ and there is no $U$’s other than $U_1$ and $U_2$.

References