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Abstract

Let $X$ be a separable metrizable coset-space of a locally compact group, which has a local cross-section and admits a nontrivial flow. Let $\mathcal{H}(X)$ be the group of homeomorphisms on $X$, endowed with the compact-open topology, and $\mathcal{H}(X, x)$ the subspace of $\mathcal{H}(X)$ consisting of those homeomorphisms which fix a point $x$ of $X$. Then $\mathcal{H}(X)$ is an $L^2$-manifold if and only if $X$ is an ANR and $\mathcal{H}(X, x)$ is an $L^2$-manifold. Among applications of this, we see that if $X$ is the plane $\mathbb{R}^2$, or punctured real projective plane, or punctured torus, then $\mathcal{H}(X)$ is an $L^2$-manifold.

Introduction

Let $\mathcal{H}(X)$ be the group of homeomorphisms on a topological space $X$, endowed with the compact-open topology. For twelve years now there has been considerable interest in the question of whether $\mathcal{H}(X)$ is an $L^2$-manifold for a manifold $X$. Among (finite-dimensional) manifolds for which the question is affirmatively answered there are metrizable connected 1-manifolds (R. D. Anderson [1], T. Karube [5]), compact metric 2-manifolds (R. Luke and W. K. Mason [7], H. Toruńczyk [9]). And it is reported that if $X$ is the plane $\mathbb{R}^2$ less an infinite closed discrete set then $\mathcal{H}(X)$ is not locally contractible (J. Keesling [6], p. 2). In this paper, for such a separable metrizable coset-space $X$ of a locally compact group that has a local cross-section and admits a nontrivial flow, it is shown that $\mathcal{H}(X)$ is an $L^2$-manifold if and only if $X$ is an ANR and $\mathcal{H}(X, x)$ is an $L^2$-manifold, where $\mathcal{H}(X, x)$ is the subspace of $\mathcal{H}(X)$ consisting of those homeomorphisms which fix a point $x$ of $X$. Joining this fact and a result of R. Arens, we see that if $X$ is a positive-dimensional compact coset-space of a Lie group then to be an $L^2$-manifold for $\mathcal{H}(X)$ and $\mathcal{H}(X-x)$ is equivalent ($x \in X$). Applying the results of R. Luke, W. K. Mason and H. Toruńczyk to this, we know that if $X$ is a compact
2-manifold homeomorphic to a coset-space of a Lie group then $\mathcal{H}(X-x)$ is an $l_4$-manifold ($x \in X$). Thus we obtain non-compact 2-manifolds whose homeomorphism groups are $l_4$-manifolds.

1. Homeomorphism groups of locally compact homogeneous spaces.

Let $G$ be a locally compact Hausdorff topological group, $X = G/H$ the left coset-space of $G$ by a closed subgroup $H$. Assume that the coset-space $X$ has a local cross-section. Let $\mathcal{H}(X)$ (resp. $\mathcal{H}(X-x)$) be the group of homeomorphisms on $X$ (resp. $X-x$, where $x \in X$), endowed with the compact-open topology, and $\mathcal{H}(X, x)$ the subspace of $\mathcal{H}(X)$ consisting of those homeomorphisms which fix a point $x \in X$. These notations $G, X, \mathcal{H}(X), \mathcal{H}(X, x)$ will keep these meanings throughout the paper.

THEOREM 1. Let $G, X, \mathcal{H}(X), \mathcal{H}(X, x)$ be those as above. Let $x_0$ be the coset containing the identity of $G$, and $p$ the projection of $\mathcal{H}(X)$ onto $X$ defined by $p(\phi) = \phi(x_0)$ ($\phi \in \mathcal{H}(X)$). Then for each $x \in X$ there exists a neighborhood $U$ of $x$ in $X$ such that $p^{-1}(U) = U \times \mathcal{H}(X, x_0)$.

(Here "-" means "homeomorphic to", and "$A \times B$" means "the product space of spaces $A$ and $B$.")

PROOF. As the space $X$ is locally compact Hausdorff, the product operation in $\mathcal{H}(X)$ is continuous and the translations in $\mathcal{H}(X)$ are homeomorphisms, while the inverse operation in $\mathcal{H}(X)$ is not necessarily continuous. Let $\pi$ be the natural projection of $G$ onto $X$, $l_a$ the left translation in $X$ by $a \in G$ defined by $l_a(x) = \pi(ab)$ ($b \in x \in X$), and $\mathcal{L} = \{l_a|a \in G\}$. $\mathcal{L}$ is a subgroup of $\mathcal{H}(X)$. Let $\omega$ be the mapping $G \to \mathcal{L}$ defined by $\omega(a) = l_a$ ($a \in G$), then $\omega$ is a continuous surjective homomorphism. For an arbitrary point $x \in X$, let $f$ be a local cross-section defined on an open neighborhood $U$ of $x$ and put $W = (\omega \circ f)(U)$. Then $p|W$ and $\omega \circ f$ are homeomorphisms $W \to U$, $U \to W$ respectively and they are inverses of each other. On the other hand let $\mathcal{H}^*$ be the coset-space $\mathcal{H}(X)/\mathcal{H}(X, x_0)$, $\pi^*$ the natural projection $\mathcal{H}(X) \to \mathcal{H}^*$, and let $q = p \circ \pi^{*-1}$. The mapping $q$ is a continuous bijection $\mathcal{H}^* \to X$ and $\pi^*|W$ is a continuous bijection $W \to q^{-1}(U)$. Thus $p^{-1}(U) = W \circ \mathcal{H}(X, x_0)$. Now let $\Phi$ be the mapping $W \times \mathcal{H}(X, x_0) \to p^{-1}(U)$ defined by $\Phi((w, \phi)) = w \circ \phi$ ($(w, \phi) \in W \times \mathcal{H}(X, x_0)$), then $\Phi$ is a continuous bijection. To show that $\Phi^{-1}$ is continuous, put $N = \ker \omega$, and let $G'$ be the factor group $G/N$, $\pi'$ the natural projection $G \to G'$, and let $x' = \pi' \circ f(x)$. We can choose a closure compact, open symmetric neighborhood $V'$ of $x'$ in $G'$ and an open neighborhood $U_1$ of $x$ in $X$ such that $f(U_1) \subset \pi'^{-1}(V')$ and $U_1 \subset U$. Put $W_1 = (\omega \circ f)(U_1)$. The mapping $\omega' = \omega \circ \pi'^{-1}$: $G' \to \mathcal{L}$ is a continuous isomorphism and its restriction on $\text{Cl} V'$ is a homeomorphism such that $W_1 \subset \omega'(\text{Cl} V')$. Since $\omega'(\text{Cl} V')$ is symmetric, the mapping $W_1 \to \omega'(\text{Cl} V')$ which maps $w$ to $\omega^{-1}(w \in W_1)$ is well-defined.
and continuous. Using the fact we can prove that $\Phi^{-1}|W \circ \mathcal{H}(X, x_\alpha)$ is continuous. Consequently

$$p^{-1}(U) = W \circ \mathcal{H}(X, x_\alpha) = U \times \mathcal{H}(X, x_\alpha).$$

**COROLLARY (J. Keesling).** If $G$ is a locally compact Hausdorff topological group, then $\mathcal{H}(G)$ is a product space over $G$.

**PROOF.** In this case we can consider in the proof of Theorem 1 that $H = \{\text{the identity}\} \subseteq G = U$, $W = \mathcal{L} = G$, and $\mathcal{L}$ is a topological group.

The fact in Corollary is found in a remark of J. Keesling [6, p.15].

Now from Theorem 1 we see that $\mathcal{H}(G)$ is locally homeomorphic to the product space $X \times \mathcal{H}(X, x)$. Thus in particular for $\mathcal{H}(G)$ to be locally contractible (or locally connected), $X$ must be so.

A space is called an $I_\alpha$-manifold if it is separable metrizable space which is locally homeomorphic to $I_\alpha$.

On the condition for $\mathcal{H}(G)$ to be an $I_\alpha$-manifold we have the following.

**THEOREM 2.** Let $X$ be a separable metrizable coset-space of a locally compact Hausdorff topological group. Assume that $X$ has a local cross-section and admits a nontrivial flow. Then $\mathcal{H}(X)$ is an $I_\alpha$-manifold if and only if $X$ is an ANR and $\mathcal{H}(X, x)$ is an $I_\alpha$-manifold for a fixed point $x \in X$.

(Here "ANR" means "absolute neighborhood retract for the class of all metrizable spaces.")

**PROOF.** Since $X$ is separable metrizable, locally compact Hausdorff, both $\mathcal{H}(X)$ and $\mathcal{H}(X, x)$ are separable metrizable. If $\mathcal{H}(X)$ is an $I_\alpha$-manifold, then $X \times \mathcal{H}(X, x)$ is an $I_\alpha$-manifold and so a local ANR, hence it is an ANR. Then both $X$ and $\mathcal{H}(X, x)$ are ANR's. By a theorem of Toruńczyk [9], $\mathcal{H}(X, x)$ is an $I_\alpha$-manifold. The converse can be easily proved using the properties of ANR's and the theorem of Toruńczyk.

2. Homeomorphism groups of locally connected, compact homogeneous spaces.

We apply Theorem 2 to $\mathcal{H}(X)$ for locally connected compact coset-spaces $X$, and show a sufficient condition for $\mathcal{H}(X)$ and $\mathcal{H}(X-x)$ to be $I_\alpha$-manifolds is equivalent, while taking account of the assumption "local connectedness", propositions hereafter can be derived basing on a McCarty's result [8, p. 295] also.

**LEMMA (R. Arens).** Let $X$ be a locally connected, compact Hausdorff space, and $x$ an arbitrary point of $X$. Then $\mathcal{H}(X, x)$ is topologically ismorphic to $\mathcal{H}(X-x)$.

**PROOF.** As McCarty [8] has remarked, this is shown by Theorems 1-3, and 4 of [3], and Theorem 2 of [2].
By Theorem 2 and Lemma we have the following.

THEOREM 3. Let $X$ be a locally connected, compact metrizable coset-space of a locally compact Hausdorff topological group. Assume that $X$ is an ANR and has a local cross-section and admits a nontrivial flow. Then $\mathcal{H}(X)$ is an $l_1$-manifold if and only if $\mathcal{H}(X-x)$ is an $l_1$-manifold where $x$ is any fixed point of $X$.

REMARK 1. A coset-space $G/H$ of a locally compact Hausdorff topological group $G$ has a local cross-section if 1) $G$ is separable metrizable finite-dimensional, or 2) there are no arbitrarily small nontrivial subgroups in $H$, or 3) all arbitrarily small subgroups of $G$ are in $H$ (T. Karube [4]).

REMARK 2. Following spaces admit a nontrivial flow: 1) a metric space that contains a subset $Y$ with nonempty interior such that $Y$ is homeomorphic to $\mathbb{R}^n$ or $I^\infty$, 2) a locally compact Hausdorff topological group which is not totally disconnected (J. Keesling [6]).

COROLLARY 1. If $X$ is a positive-dimensional compact coset-space of a Lie group, then the same conclusion as in Theorem 3 holds.

EXAMPLES. The following spaces satisfy the condition on $X$ in Corollary 1: torus $T^n$, sphere $S^n$, projective space $RP_n$, $CP_n$, or $HP_n$ over the field of real numbers, complex numbers, or quaternions respectively ($n > 0$) --- $S^n = O(n+1)/O(n)$, $RP_n = O(n+1)/(O(n) \times S^n)$, $CP_n = U(n+1)/(U(n) \times S^n)$, $HP_n = Sp(n+1)/(Sp(n) \times S^n)$. ($O(n)$: the orthogonal group, $U(n)$: the unitary group, $Sp(n)$: the sympletic group)

COROLLARY 2. If $X$ is a compact 2-manifold homeomorphic to a coset-space of a Lie group, then $\mathcal{H}(X-x)$ is an $l_1$-manifold for each point $x$ of $X$.

PROOF. By a theorem of Toruńczyk [9], for a separable metric space $M$ which admits a nontrivial flow, $\mathcal{H}(M)$ is an $l_1$-manifold if and only if it is an ANR. When $M$ is a compact metric 2-manifold without boundary, Luke and Mason [7] have shown that $\mathcal{H}(M)$ is an ANR. Hence by Corollary 1 we see that $\mathcal{H}(X-x)$ is an $l_1$-manifold.

EXAMPLES. The following spaces satisfy the condition on $X-x$ in Corollary 2: plane $\mathbb{R}^2$, punctured torus, punctured real projective plane, etc.

REMARK. We can treat analogously “local contractibility” in place of “$l_1$-manifold”. Therefore if $X$ is a compact coset-space of a Lie group, then $\mathcal{H}(X-x)$ is locally contractible, by a theorem of A. V. Černavskiĭ (Math. USSR Sb. 8 (1969)). Thus, for example, if $Y$ is a Euclidean space, or a punctured torus, or a punctured projective space over the field of real numbers, complex numbers, or quaternions, then $\mathcal{H}(Y)$ is locally contractible.
References