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Homeomorphism Groups of Homogeneous Spaces

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Abstract

Let X be a separable metrizable coset-space of a locally compact group, which has a local cross-section and admits a nontrivial flow. Let $\mathscr{H}(X)$ be the group of homeomorphisms on X, endowed with the compact-open topology, and $\mathscr{H}(X, x)$ the subspace of $\mathscr{H}(X)$ consisting of those homeomorphisms which fix a point x of X. Then $\mathscr{H}(X)$ is an l_2-manifold if and only if X is an ANR and $\mathscr{H}(X, x)$ is an l_2-manifold. Among applications of this, we see that if X is the plane R^2, or punctured real projective plane, or punctured torus, then $\mathscr{H}(X)$ is an l_2-manifold.

Introduction

Let $\mathscr{H}(X)$ be the group of homeomorphisms on a topological space X, endowed with the compact-open topology. For twelve years now there has been considerable interest in the question of whether $\mathscr{H}(X)$ is an l_2-manifold for a manifold X. Among (finite-dimensional) manifolds for which the question is affirmatively answered there are metrizable connected 1-manifolds (R. D. Anderson [1], T. Karube [5]), compact metric 2-manifolds (R. Luke and W. K. Mason [7], H. Toruńczyk [9]). And it is reported that if X is the plane R^2 less an infinite closed discrete set then $\mathscr{H}(X)$ is not locally contractible (J. Keesling [6], p. 2). In this paper, for such a separable metrizable coset-space X of a locally compact group that has a local cross-section and admits a nontrivial flow, it is shown that $\mathscr{H}(X)$ is an l_2-manifold if and only if X is an ANR and $\mathscr{H}(X, x)$ is an l_2-manifold, where $\mathscr{H}(X, x)$ is the subspace of $\mathscr{H}(X)$ consisting of those homeomorphisms which fix a point x of X. Joining this fact and a result of R. Arens, we see that if X is a positive-dimensional compact coset-space of a Lie group then to be an l_2-manifold for $\mathscr{H}(X)$ and $\mathscr{H}(X-x)$ is equivalent ($x \in X$). Applying the results of R. Luke, W. K. Mason and H. Toruńczyk to this, we know that if X is a compact

2-manifold homeomorphic to a coset-space of a Lie group then $\mathscr{H}(X-x)$ is an l_2-manifold $(x\in X)$. Thus we obtain non-compact 2-manifolds whose homeomorphism groups are l_2-manifolds.

1. Homeomorphism groups of locally compact homogeneous spaces.

Let G be a locally compact Hausdorff topological group, $X=G/H$ the left coset-space of G by a closed subgroup H. Assume that the coset-space X has a local cross-section. Let $\mathscr{H}(X)$ (resp. $\mathscr{H}(X-x)$) be the group of homeomorphisms on X (resp. $X-x$, where $x\in X$), endowed with the compact-open topology, and $\mathscr{H}(X, x)$ the subspace of $\mathscr{H}(X)$ consisting of those homeomorphisms which fix a point $x\in X$. These notations G, X, $\mathscr{H}(X)$, $\mathscr{H}(X,x)$, $\mathscr{H}(X-x)$ will keep these meanings throughout the paper.

THEOREM 1. *Let G, X, $\mathscr{H}(X)$, $\mathscr{H}(X,x)$ be those as above. Let x_0 be the coset containing the identity of G, and p the projection of $\mathscr{H}(X)$ onto X defined by $p(\phi)=\phi(x_0)$ $(\phi\in\mathscr{H}(X))$. Then for each $x\in X$ there exists a neighborhood U of x in X such that $p^{-1}(U)\approx U\times\mathscr{H}(X, x_0)$.*

(Here "$\approx$" means "homeomorphic to ", and "$A\times B$" means "the product space of spaces A and B.)

PROOF. As the space X is locally compact Hausdorff, the product operation in $\mathscr{H}(X)$ is continuous and the translations in $\mathscr{H}(X)$ are homeomorphisms, while the inverse operation in $\mathscr{H}(X)$ is not necessarily continuous. Let π be the natural projection of G onto X, l_a the left translation in X by $a\in G$ defined by $l_a(x)=\pi(ab)$ $(b\in x\in X)$, and $\mathscr{L}=\{l_a\mid a\in G\}$. $\mathscr{L}$ is a subgroup of $\mathscr{H}(X)$. Let ω be the mapping $G\to\mathscr{L}$ defined by $\omega(a)=l_a$ $(a\in G)$, then ω is a continuous surjective homomorphism. For an arbitrary point $x\in X$, let f be a local cross-section defined on an open neighborhood U of x and put $W=(\omega\circ f)(U)$. Then $p|W$ and $\omega\circ f$ are homeomorphisms $W\to U$, $U\to W$ respectively and they are inverses of each other. On the other hand let $\mathscr{H}^*$ be the coset-space $\mathscr{H}(X)/\mathscr{H}(X,x_0)$, π^* the natural projection $\mathscr{H}(X)\to\mathscr{H}^*$, and let $q=p\circ\pi^{*-1}$. The mapping q is a continuous bijection $\mathscr{H}^*\to X$ and $\pi^*|W$ is a continuous bijection $W\to q^{-1}(U)$. Thus $p^{-1}(U)=W\circ\mathscr{H}(X,x_0)$. Now let Φ be the mapping $W\times\mathscr{H}(X,x_0)\to p^{-1}(U)$ defined by $\Phi((w,\phi))=w\circ\phi$ $((w,\phi)\in W\times\mathscr{H}(X,x_0))$, then Φ is a continuous bijection. To show that Φ^{-1} is continuous, put $N=\ker\omega$, and let G' be the factor group G/N, π' the natural projection $G\to G'$, and let $x'=\pi'\circ f(x)$. We can choose a closure compact, open symmetric neighborhood V' of x' in G' and an open neighborhood U_1 of x in X such that $f(U_1)\subset\pi'^{-1}(V')$ and $U_1\subset U$. Put $W_1=(\omega\circ f)(U_1)$. The mapping $\omega'=\omega\circ\pi'^{-1}$: $G'\to\mathscr{L}$ is a continuous isomorphism and its restriction on $\mathrm{Cl}\,V'$ is a homeomorphism such that $W_1\subset\omega'(\mathrm{Cl}\,V')$. Since $\omega'(\mathrm{Cl}\,V')$ is symmetric, the mapping $W_1\to\omega'(\mathrm{Cl}\,V')$ which maps w to w^{-1} $(w\in W_1)$ is well-defined and continuous. Using the fact we can prove that $\Phi^{-1}|W_1 \circ \mathscr{H}(X, x_0)$ is continuous. Consequently

$p^{-1}(U_1) = W_1 \circ \mathscr{H}(X, x_0) \approx U_1 \times \mathscr{H}(X, x_0)$.

COROLLARY (J. Keesling). *If G is a locally compact Hausdorff topological group, then $\mathscr{H}(G)$ is a product space over G.*

PROOF. In this case we can consider in the proof of Theorem 1 that $H=$ {the identity}, $X=G=U$, $W=\mathscr{L}\approx G$, and $\mathscr{L}$ is a topological group.

The fact in Corollary is found in a remark of J. Keesling [6, p.15].

Now from Theorem 1 we see that $\mathscr{H}(X)$ is locally homeomorphic to the product space $X \times \mathscr{H}(X, x)$. Thus in particular for $\mathscr{H}(X)$ to be locally contractible (or locally connected), X must be so.

A space is called an *l_2-manifold* if it is separable metrizable space which is locally homeomorphic to l_2.

On the condition for $\mathscr{H}(X)$ to be an l_2-manifold we have the following.

THEOREM 2. *Let X be a separable metrizable coset-space of a locally compact Hausdorff topological group. Assume that X has a local cross-section and admits a nontrivial flow. Then $\mathscr{H}(X)$ is an l_2-manifold if and only if X is an ANR and $\mathscr{H}(X, x)$ is an l_2-manifold for a fixed point $x \in X$.*

(Here "ANR" means" absolute neighborhood retract for the class of all metrizable spaces.)

PROOF. Since X is separable metrizable, locally compact Hausdorff, both $\mathscr{H}(X)$ and $\mathscr{H}(X, x)$ are separable metrizable. If $\mathscr{H}(X)$ is an l_2-manifold, then $X \times \mathscr{H}(X, x)$ is an l_2-manifold and so a local ANR, hence it is an ANR. Then both X and $\mathscr{H}(X, x)$ are ANR's. By a theorem of Toruńczyk [9], $\mathscr{H}(X, x)$ is an l_2-manifold. The converse can be easily proved using the properties of ANR's and the theorem of Toruńczyk.

2, Homeomorphism groups of locally connected. compact homogeneous spaces.

We apply Theorem 2 to $\mathscr{H}(X)$ for locally connected compact coset-spaces X, and show a sufficient condition for $\mathscr{H}(X)$ and $\mathscr{H}(X-x)$ to be l_2-manifolds is equivalent, while taking accunt of the assumption "local connectedness", propositions hereafter can be derived basing on a McCarty's result [8, p. 295] also.

LEMMA (R. Arens). *Let X be a locally connected, compact Hausdorff space, and x an arbitrary point of X. Then $\mathscr{H}(X, x)$ is topologically ismorphic to $\mathscr{H}(X-x)$.*

PROOF. As McCarty [8] has remarked, this is shown by Theorems 1 3, and 4 of [3], and Theorem 2 of [2].

By Theorem 2 and Lemma we have the following.

THEOREM 3. *Let X be a locally connected, compact metrizable coset-space of a locally compact Hausdorff topological group. Assume that X is an ANR and has a local cross-section and admits a nontrivial flow. Then $\mathscr{H}(X)$ is an l_2-manifold if and only if $\mathscr{H}(X-x)$ is an l_2-manifold where x is any fixed point of X.*

REMARK 1. A coset-space G/H of a locally compact Hausdorff topological group G has a local cross-section if 1) G is separable metrizable finite-dimensional, or 2) there are no arbitrarily small nontrivial subgroups in H, or 3) all arbitrarily small subgroups of G are in H (T. Karube [4]).

REMARK 2. Following space admit a nontrivial flow : 1) a metric space that contains a subset Y with nonempty interior such that Y is homeomrphic to R^n or l_2 or I^∞, 2) a locally compact Hausdorff topological group which is not totally disconnected (J. Keesling [6]).

COROLLARY 1. *If X is a positive-dimensional compact coset-space of a Lie group, then the same conclusion as in Theorem 3 holds.*

EXAMPLES. The following spaces satisfy the condition on X in Corollary 1 : torus T^n, sphere S^n, projective space $\mathsf{R}P_n$, $\mathsf{C}P_n$, or $\mathsf{H}P_n$ over the field of real numbers, complex numbers, or quaternions respectively $(n>0)$ —— $S^n \approx O(n+1)/O(n)$, $\mathsf{R}P_n \approx O(n+1)/(O(n)\times S^0)$, $\mathsf{C}P_n \approx U(n+1)/(U(n)\times S^1)$, $\mathsf{H}P_n \approx Sp(n+1)/(Sp(n)\times S^3)$. ($O(n)$: the orthogonal group, $U(n)$: the unitary group, $Sp(n)$: the sympletic group)

COROLLARY 2. *If X is a compact 2-manifold homeomorphic to a coset-space of a Lie group, then $\mathscr{H}(X-x)$ is an l_2-manifold for each point x of X.*

PROOF. By a theorem of Toruńczyk [9], for a separable metric space M which admits a nontrivial flow, $\mathscr{H}(M)$ is an l_2-manifld if and only if it is an ANR. When M is a compact metric 2-manifold without boundary, Luke and Mason [7] have shown that $\mathscr{H}(M)$ is an ANR. Hence by Corollary 1 we see that $\mathscr{H}(X-x)$ is an l_2-manifold.

EXAMPLES. The following spaces satisfy the condition on $X-x$ in Corollary 2 : plane R^2, punctured torus, punctured real projective plane, etc.

REMARK. We can treat analogously "local contractibility" in place of "l_2-manifold". Therefore if X is a compct coset-space of a Lie group, then $\mathscr{H}(X-x)$ is locally contractible, by a theorem of A.V. Černavskiĭ (Math. USSR Sb. **8** (1969)). Thus, for example, if Y is a Euclidean space, or a punctured torus, or a punctured projective space over the field of real numbers, complex numbers, or quarternions, then $\mathscr{H}(Y)$ is locally contractible.

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