Hilbert subgroups of the Full Homeomorphism Group of a Topological Group

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Abstract

Let \( \mathcal{H} \) be the full homeomorphism group (with the compact-open topology) of a finite-dimensional connected locally compact nontrivial topological group. Then we can actually construct in \( \mathcal{H} \) a closed subgroup intersecting each coset of the automorphism group at most one point. The subgroup is homeomorphic to the infinite-dimensional separable Hilbert space but is not complete with respect to the uniformity of compact convergence defined by the right uniformity of the base group. The similar results hold for the full homeomorphism group of a Hausdorff space that is locally Euclidean around at least one point.

Introduction

Let \( X \) be a metric space, and \( \mathcal{H} \) the full homeomorphism group of \( X \) with the compact-open topology. The main root to prevent us from free use of the theory of topological transformation groups for \( \mathcal{H} \) is the existence of a subspace that is infinite-dimensional and not locally compact. J. Keesling [6, Th. III.2, p.6] has shown the existence of Hilbert factors of \( \mathcal{H} \) for a metric space \( X \) which admits a nontrivial flow. The purpose of the present paper is to construct actually in \( \mathcal{H} \) a closed subgroup which is homeomorphic to the infinite-dimensional separable Hilbert space \( \ell_2 \), mainly in the case where the base space \( X \) is a topological group.

In \$1\$, a subgroup of the above character is constructed for a Hausdorff space \( X \) at least one point of which has an open neighborhood homeomorphic to an open subset of a Euclidean half-space (Theorem 1.5).

In \$2\$, basic properties of several important subgroups of the full homeomorphism group \( \mathcal{H} \) of a topological group \( G \) are treated. Let \( \mathcal{A} \) be the automorphism group of \( G \), and \( \mathcal{I} \) the group of homeomorphisms which leave the identity element of \( G \) fixed. Then the investigation of topologies of \( \mathcal{H} \) is mainly reduced to that of crossing sets of cosets of \( \mathcal{A} \) in \( \mathcal{I} \).

In \$3\$, in the case where \( G \) is a finite-dimensional connected locally compact nontrivial topological group, we construct in \( \mathcal{H} \) a closed subgroup homeomorphic to \( \ell_2 \).
and crossing cosets of $\mathcal{A}$ in $\mathcal{F}$ (Theorem 3.3).

Throughout the paper, the topology used for any mapping family is the compact-open topology, and topological groups are $T_0$ topological groups and so completely regular.

§1. Hilbert subgroups of the full homeomorphism group of a topological space.

We prepare three lemmas whose proofs are straightforward.

**Lemma 1.1.** Let $B$ be the closed unit ball (or half-ball) centered at origin $o$ in Cartesian $n$-space, with a radius $I$, and $\mathcal{H}^+(I)$ the group of all orientation preserving homeomorphisms of $I$ onto itself. For each $f \in \mathcal{H}^+(I)$, let $u$ be the mapping of $B$ onto itself defined as follows:

$$u(x) = \frac{f(\|x\|)}{\|x\|}x \text{ if } x \in B-o, \text{ and } u(o) = o.$$  

Let $\mathcal{G}$ be the set of all mappings $u$ associated with $f \in \mathcal{H}^+(I)$ as above, and let $\omega$ be the associating mapping. Then $\mathcal{G}$ with the compact-open topology is a topological transformation group of $B$ each element of which leaves the origin and every boundary point of $B$ fixed, and is isomorphic to $\mathcal{H}^+(I)$ as topological groups by the mapping $\omega$.

**Definition 1.2.** A topological space $X$ has the property (P) if the full homeomorphism group of $X$ becomes a topological transformation group acting on $X$ under the compact-open topology.

For example, the following spaces have the property (P) : i) compact uniform spaces ([3], the remark after Th.1, p.663), and ii) locally connected, locally compact Hausdorff spaces ([2], Th.4, p.598).

**Lemma 1.3.** Let $X$ be a topological space, $A$ its closed subset having the property (P), and $\mathcal{G}$ a group of homeomorphisms of $A$ onto itself each of which keeps every boundary point of $A$ fixed. For each $f \in \mathcal{G}$, let $f^*$ be the mapping of $X$ onto itself as follows:

$$f^*(x) = f(x) \text{ if } x \in A \text{ and } f^*(x) = x \text{ if } x \in X-A.$$  

Let $\mathcal{G}^*$ be the set of all $f^*$ associated with $f \in \mathcal{G}$ as above, and let $\omega$ be the associating mapping. Then $\mathcal{G}$ with the compact-open topologies are topological transformation groups of $A$ and $X$ respectively, and isomorphic as topological groups by the mapping $\omega$.

**Lemma 1.4.** Let $X$ and $Y$ be two topological spaces, and $X'$ and $Y'$ be subspaces of $X$ and $Y$ respectively. Let $\mathcal{G}(X;Y)$ be the family of all continuous mappings of $X$ into $Y$, with compact-open topology, and let $\mathcal{G}(X';Y)$ and $\mathcal{G}(X';Y')$ denote the families defined as well. Then

i) the mapping $f \mapsto f \upharpoonright X'$ of $\mathcal{G}(X;Y)$ into $\mathcal{G}(X';Y)$ is continuous, and

ii) if $\mathcal{G}$ denotes the family of all continuous mappings of $X$ into $Y$ which map $X'$ into $Y'$ then the mapping $f \mapsto f \upharpoonright X'$ of $\mathcal{G}$ into $\mathcal{G}(X';Y')$ is continuous.

**Theorem 1.5.** Let $X$ be a Hausdorff space at least one point $p$ of which has an open neighborhood $V$ homeomorphic to an open subset of a Euclidean half-space, and $\mathcal{H}$ be the full homeomorphism group of $X$ with the compact-open topology. Then $\mathcal{H}$ has a closed subgroup $\mathcal{G}$ that is a topological group isomorphic to the group $\mathcal{H}^+(I)$ of all
orientation preserving homeomorphisms of the closed interval \([0,1]\). Thus \(\mathcal{G}\) is homeomorphic to the infinite-dimensional separable Hilbert space.

PROOF. For simplicity suppose that the neighborhood \(V\) is homeomorphic to an open subset of a Euclidean space \(\mathbb{R}^n\) — the case where the image of the point \(p\) is a boundary point of a Euclidean half-space can be treated as well. Let \(B^*\) be the closed unit ball centered at origin in \(\mathbb{R}^n\), with a radius \(l^*\). There is a homeomorphism of \(B^*\) into \(V\) mapping the origin to the point \(p\). Let \(B\) and \(I\) be the image of \(B^*\) and \(I^*\) respectively by the homeomorphism. Hereafter, for simplicity, we consider \(B\) is identical with \(B^*\) and \(I\) with the closed unit interval \([0,1]\). As in Lemma 1.1 we can construct a homeomorphism group \(\mathcal{G}\) of \(B\) which is isomorphic to \(\mathcal{H}^+(I)\) and leave the point \(p\) and every boundary point of \(B\) fixed. Associate each \(u \in \mathcal{G}\) with the following mapping \(v\) of \(X\) onto itself:

\[ v(x) = \begin{cases} u(x) & \text{if } x \in B, \\ x & \text{if } x \in X - B. \end{cases} \]

Let \(\mathcal{G}\) be the set of all \(v\) associated with \(u \in \mathcal{G}\). Then \(\mathcal{G}\) is a topological transformation group of \(X\), isomorphic to \(\mathcal{H}^+(I)\) by Lemmas 1.1 and 1.3.

Next we show that \(\mathcal{G}\) is closed in \(\mathcal{H}\). Let \(\pi\) be the mapping \(x \mapsto \|x\|\) of \(B\) onto \(I\). Suppose that a net \((v_\lambda; \lambda \in \Lambda)\) in \(\mathcal{G}\) converges to \(w \in \mathcal{H}\), and each \(v_\lambda\) is defined by \(f_\lambda \in \mathcal{H}^+(I)\) and \(u_\lambda \in \mathcal{G}\) successively. Then

\[ f_\lambda \pi = \pi u_\lambda \quad \text{on } B, \quad v_\lambda \mid B = u_\lambda. \]

Since \(v_\lambda\) converges to \(w\) in the space \(\mathcal{G}(X; X)\) of all continuous mappings of \(X\) into itself also,

we have by i) of Lemma 1.4,

\[ u_\lambda \mid B \text{ converges to } w \mid B \text{ in } \mathcal{G}(B; X). \]

Since \(u_\lambda(B) = B\) and \(B\) is closed, we have \((w \mid B)(B) \subset B\), so that by ii) of Lemma 1.4,

\[ u_\lambda \mid B \text{ converges to } w \mid B \text{ in } \mathcal{G}(B; B). \]

Hence \(f_\lambda \pi = \pi u_\lambda \) converges to \(\pi w \mid B \) in \(\mathcal{G}(B; B)\), and \(f_\lambda \mid I\) converges to \(\pi w \mid I\) in \(\mathcal{G}(I; B)\).

Since \(f_\lambda(I) = I\) and \(I\) is closed, we have

\[ (\pi w \mid I)(I) = I, \]

so that by ii) of Lemma 1.4,

\[ f_\lambda \mid I \text{ converges to } \pi w \mid I \text{ in } \mathcal{G}(I; I). \]

Define the mapping \(f^*\) of \(I\) into itself as follows:

\[ f^*(x) = \pi w(x) \quad (x \in I) \]

Then it is easy to see that the mapping \(f^*\) is an orientation preserving homeomorphism of \(I\) onto itself. Starting from \(f^* \in \mathcal{H}^+(I)\), define \(u^*\) and \(v^*\) as well as the construction of \(v\). Then by (1), (2), and (3), we have

\[ u^* = w \mid B, \quad \text{and } v^* = w. \]

Therefore \(w\) is an element of \(\mathcal{G}\).

It is reported R. D. Anderson [1] has proved that \(\mathcal{H}^+(I)\) is homeomorphic to the infinite-dimensional separable Hilbert space, an easy proof of which is found in a paper of J. Keesling [6, Th. III. 1, p. 5].
COROLLARY 1.6. Let $X$, $\mathcal{H}$, and $\mathcal{Q}$ be as in Theorem 1.5. Then both $\mathcal{H}$ and $\mathcal{Q}$ are not locally compact, not closed in the space $\mathcal{C}(X)$ of all continuous mappings of $X$ into itself, and not complete with respect to the uniformity of compact convergence if $X$ is a uniform space moreover.

PROOF. There exists a net $(f_{i})_{0<\lambda<1}$ in $\mathcal{H}^{+}(I)$ which converges in $\mathcal{C}(I)$ to a continuous but non-homeomorphic mapping $f_0$ of $I$ onto itself as $\lambda \to 0$ (see Example 1.7 below). Starting from the mapping $f_{i}$ ($0 \leq \mu < 1$), we can define continuous mappings $u_{\pi}$ and $v_{\pi}$ in the same way as in the proof of Theorem 1.5. Then $(u_{i})_{0<\lambda<1}$ is a net in $\mathcal{Q}$, which converges to $v_{0} \in \mathcal{C}(X) - \mathcal{H}$. Hence $\mathcal{Q}$ is not closed in $\mathcal{C}(X)$, and so does $\mathcal{H}$. Thus both $\mathcal{H}$ and $\mathcal{Q}$ are not complete with respect to the uniformity of compact convergence. On the other hand from the fact that $\mathcal{Q}$ is a closed subgroup of $\mathcal{H}$ which is homeomorphic to the Hilbert space $\ell_{2}$, we see that $\mathcal{Q}$ is not locally compact, and so $\mathcal{H}$ is not locally compact.

EXAMPLE 1.7. Let $I$ be the closed interval $[0,1]$. For each $\lambda \in I$, define the continuous mapping $f_{\lambda}$ of $I$ onto itself as follows:

$$f_{\lambda}(t) = \begin{cases} 2\lambda t & \text{if } 0 \leq t \leq 1/2 \\ (1-\lambda)(2t-1)+\lambda & \text{if } 1/2 < t \leq 1 \end{cases}$$

Then $f_{\lambda} \in \mathcal{H}^{+}(I)$ if $0 < \lambda < 1$, and $f_{0}, f_{1} \in \mathcal{C}(I) - \mathcal{H}(I)$, where $\mathcal{H}(I)$ is the full homeomorphism group of $I$. Under the topology of simple-, compact-, and uniform-convergence, $f_{\lambda}$ converges to $f_{0}$ (resp. $f_{1}$) if $\lambda \to 0$ (resp. $\lambda \to 1$).

§ 2 Subgroups of the full homeomorphism group of a topological group.

Let $G$ be a $T_{0}$ topological group. We use the following notations.

- $\mathcal{H}$: The group of all homeomorphisms of $G$ onto itself.
- $\mathcal{J}$: The group of all bicontinuous automorphisms of $G$.
- $\mathcal{F}$: The group of all inner automorphisms of $G$.
- $\mathcal{R}$: The group of all right translations of $G$.
- $\mathcal{L}$: The group of all left translations of $G$.
- $\mathcal{I}$: The group of all homeomorphisms of $G$ onto itself, which leave the identity element $e$ of $G$ fixed, i.e. the isotropy group at $e$.
- $\mathcal{N}$: The normalizer of $\mathcal{R}$ in $\mathcal{H}$.
- $\mathcal{N}'$: The normalizer of $\mathcal{Q}$ in $\mathcal{H}$.
- $\mathcal{E}$: The identity mapping of $G$ onto itself.

In this section we summarize the fundamental properties and relations of these groups, some of them are well-known.

Relations of inclusion:

1) $\mathcal{R} \cap \mathcal{F} = \{e\}$.  \hspace{1cm} 1)' \mathcal{Q} \cap \mathcal{F} = \{e\}.
2) $\mathcal{R} \cap \mathcal{J} = \mathcal{J}$.  \hspace{1cm} 2)' $\mathcal{L} \cap \mathcal{J} = \mathcal{J}$.
3) $\mathcal{H} = \mathcal{R} = \mathcal{J}$.  \hspace{1cm} 3)' $\mathcal{H} = \mathcal{L} = \mathcal{J}$.
4) $\mathcal{N} = \mathcal{R} = \mathcal{J}$.  \hspace{1cm} 4)' $\mathcal{N}' = \mathcal{L} = \mathcal{J}$.
5) $\mathcal{R} \mathcal{L} = \mathcal{L} \mathcal{R} = \mathcal{R} = \mathcal{I} = \mathcal{L} = \mathcal{J}$. 

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6) \( \mathcal{R} \mathcal{A} = \mathcal{L} \mathcal{A} \).
7) \( \mathcal{R} \mathcal{L} \cap \mathcal{A} = \mathcal{I} \).

PROOF. Both 1) and 2) are evident. For 3), note that any \( u \in \mathcal{H} \) can be written in the form
\[
u = r_{u(e)} \circ r_{u(e)}^{-1} \circ u = (u \circ r_{u(e)}) \circ r_{u(e)}^{-1},\]
where \( r_{u(e)} \) is the right translation of \( G \) by \( u(e) \) and \( v = u^{-1} \). For 4), let \( \mathcal{N}^* \) be the normalizer of \( \mathcal{R} \) in the group of all permutations of \( G \) and let \( \mathcal{A}^* \) be the group of all (algebraic) automorphisms of \( G \). Then note that \( \mathcal{N}^* = \mathcal{N} \mathcal{R} \) (cf., e.g., A. G. Kurosch (7), p. 92), and so \( \mathcal{N}^* = \mathcal{N} \mathcal{R} \cap \mathcal{H} \subseteq \mathcal{A} \mathcal{R} \). Relations 1'-4' are dual to 1)-4). Relations 5), 6), and 7) are proved directly.

To be a normal subgroup. By definition both \( \mathcal{R} \) and \( \mathcal{L} \) are normal subgroups of \( \mathcal{N} \), and so normal subgroups of \( \mathcal{R} \mathcal{L} \) also. \( \mathcal{R} \cap \mathcal{L} \) is a normal subgroup of \( \mathcal{N} \), and so \( \mathcal{R} \mathcal{L} \cap \mathcal{L} \) is a normal subgroup of \( \mathcal{R} \), \( \mathcal{L} \), and \( \mathcal{R} \mathcal{L} \) respectively. Clearly \( \mathcal{R} \mathcal{L} \) is a normal subgroup of \( \mathcal{N} \), and \( \mathcal{I} \) is a normal subgroup of \( \mathcal{N} \). In each of the remaining nontrivial cases, to be a normal subgroup is not necessarily valid, even if \( G \) is a connected Lie group.

EXAMPLE 2.1. Let \( G \) be the real vector group \( \mathbb{R}^n \), then \( \mathcal{N} \) is \( \text{GL}(n, \mathbb{R}) \) and \( \mathcal{N} \) is the group of all (non-singular) affine transformations of \( \mathbb{R}^n \). Let \( B \) be the closed unit ball centered at origin in \( \mathbb{R}^n \), and let \( u \) be a transformation of \( \mathbb{R}^n \) which leaves each exterior point of \( B \) fixed and maps some segment in \( B \) to an arc that is not a segment. Let \( r_a \) be the translation in \( \mathbb{R}^n \) by \( a \), where \( ||a|| > 2 \). Then
\[
u = r_{u(e)} \in \mathcal{H} \mathcal{N}, \quad r_a \in \mathcal{R} \cap \mathcal{L} = \mathcal{R} = \mathcal{L} = \mathcal{R} \mathcal{L}, \quad \text{and} \quad u^{-1} r_a u \in \mathcal{H} \mathcal{N}.
\]
Therefore we see that every one of the groups \( \mathcal{R} \mathcal{L} \), \( \mathcal{R} \), \( \mathcal{L} \), \( \mathcal{R} \mathcal{L} \), and \( \mathcal{N} \) is not necessarily a normal subgroup of \( \mathcal{H} \).

EXAMPLE 2.2. Let \( G \) be a non-abelian matrix group as follows:
\[
G = \left\{ (x,y,z) \mid x,y,z \in \mathbb{R} \right\}, \quad \text{where} \quad (x,y,z) = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}
\]
Under the usual topology of \( \mathbb{R}^n \), \( G \) is a 3-dimensional connected Lie group. Let \( u \) and \( v \) be the mappings of \( G \) onto itself defined by
\[
u(x,y,z) = (x+1,y,z), \quad \text{and} \quad v(x,y,z) = (x,y,z)(0,0,1) = (x,x+y,z).
\]
Then \( u \) is the right translation in \( G \) by \( (1,0,0) \), \( v \in \mathcal{I} \), and
\[
u^{-1} u v (x,y,z) = (x,x+y+1,z).
\]
Therefore the fact that \( v \in \mathcal{I} \subseteq \mathcal{A} \subseteq \mathcal{I} \), \( u \in \mathcal{H} \), and \( u^{-1} v u \in \mathcal{H} - \mathcal{I} \) implies each of the groups \( \mathcal{I} \), \( \mathcal{A} \), and \( \mathcal{I} \) is not a normal subgroup of \( \mathcal{H} \);
the fact that \( v \in \mathcal{I} \subseteq \mathcal{A} \), \( u \in \mathcal{H} \), and \( u^{-1} v u \in \mathcal{H} - \mathcal{A} \subseteq \mathcal{N} - \mathcal{I} \) implies neither \( \mathcal{I} \) nor \( \mathcal{A} \) is a normal subgroup of \( \mathcal{H} \);
and the fact that \( v \in \mathcal{I} \), \( u \in \mathcal{R} \mathcal{L} \), and \( u^{-1} v u \in \mathcal{R} \mathcal{L} - \mathcal{I} \) implies
I is not a normal subgroup of \( RL \).

Next let \( w \) be the mapping of \( G \) onto itself defined by
\[
w(x, y, z) = (x + y, y, -z).
\]
Then both \( w \) and \( w^{-1} \) belong to \( I \). Therefore we see that
neither \( I \) nor \( A \) is a normal subgroup of \( IL \).

**Closedness.** From here on throughout the paper it is supposed that \( G \) is a locally compact \( T_0 \) topological group and the compact-open topology is given on \( H \). \( I \) is inverse-isomorphic to \( G \), and \( J \) is isomorphic to \( G \) as topological groups. Both \( I \) and \( J \) are closed in \( H \), and so \( I \cap J \) is closed in \( H \). It is easy to see that both \( I \) and \( J \) are closed in \( H \). \( J \) is closed in \( H \) if \( G \) is (locally compact) locally connected or compact. In fact, let \( f \) be the mapping \( u \to uru^{-1} \) of \( H \) into itself for a fixed \( r \in H \), then \( f \) is continuous. Hence
\[
J = \bigcap_{r \in H} f^{-1}(I) \cap \bigcap_{r \in H} f^{-1}(J) = \bigcap_{r \in H} f^{-1}(I \cap J)
\]
is closed in \( H \). Moreover it is easy to see that \( I \) is closed in \( RL \), since \( I = RL \cap J \) and \( I \) is closed in \( H \). But \( I \) is not necessarily closed in \( J \), even if \( G \) is a connected Lie group. This fact was remarked by K. Nomizu and M. Gotô [9]. We can give an example to show it using an example of W. T. Est [4, p. 326].

**EXAMPLE 2.3.** Let \( G^* \) and \( G \) be matric groups as follows:
\[
G^* = \{(s, t, \alpha, \beta) \mid s, t \in R; \alpha, \beta \in C\}, \text{ where}
\[
(s, t, \alpha, \beta) = \begin{bmatrix}
e^t & 0 & \alpha \\
0 & e^t & \beta \\
0 & 0 & 1
\end{bmatrix}
\]
\[
G = \{(s, \sqrt{2}t, \alpha, \beta) \mid s \in R; \alpha, \beta \in C\}.
\]
By the usual topologies, \( G^* \) and \( G \) become connected Lie groups, and \( G \) is a dense proper normal subgroup of \( G^* \). There exist a sequence \( (g_j) \) in \( G \), converging to an element \( g^* \in G^* - G \). Let \( g_j \) and \( g^* \) be
\[
g_j = (s_j, \sqrt{2}s_j, \alpha, \beta), \ g^* = (s^*, t^*, \gamma, \delta).
\]
Let \( u_j \) and \( u^* \) be the mappings of \( G \) onto itself defined by
\[
u_j(x) = g_jxg_j^{-1} \text{ and } u^*(x) = g^*xg^*^{-1} (x \in G).
\]
Then \( u_j \in I \), \( u^* \in J - I \), and \( u_j \) converges to \( u^* \) under the topology \( \tau_p \) of pointwise convergence. It is shown below that the topology \( \tau_p \) coincides with the compact-open topology \( \tau_e \) on \( A \). In fact by a theorem of D.H.Lee and T.-S.Wu [8] \( A \) is a Lie group under the topology \( \tau \) usually used for automorphism groups. Since \( G \) is locally compact and locally connected, the topology \( \tau \) coincides with the compact-open topology \( \tau_e \) on \( A \). And so \( A \) has an equicontinuous neighborhood \( W \) of the identity \( E \) by the theorem of Ascoli. From the fact that \( \tau_p = \tau_e \) on \( W \), it follows that \( \tau_p = \tau_e \) on \( A \).

Now using the facts already shown, it is easy to see that even if \( G \) is a connected Lie group, \( I \) is not necessarily closed in \( J \), \( J \), and \( H \) respectively; and so \( RL \) is not necessarily closed in \( J \) and \( H \) respectively. While if \( G \) is "compact" or "locally compact abelian", then \( RL \) and \( I \) are closed in \( H \).
To be a semi-direct product or a direct product. In general, \( \mathcal{R} \) is not the semi-direct product of \( \mathcal{R} \) and \( \mathcal{P} \), for we have only to consider the case where \( G \) is abelian. As for \( \mathcal{A} \), it is algebraically the semi-direct product of its normal subgroup \( \mathcal{R} \) and its subgroup \( \mathcal{P} \). Let \( \omega \) be the mapping \( (r,u) \rightarrow ru \) of \( \mathcal{R} \times \mathcal{P} \) onto \( \mathcal{A} \), then \( \omega \) is a continuous bijection if \( G \) is locally compact. If \( G \) is compact, then \( \omega \) is a homeomorphism and so \( \mathcal{A} \) is the topological semi-direct product of its closed normal subgroup \( \mathcal{R} \) and its closed subgroup \( \mathcal{P} \). Also in the case where \( G \) is the \( n \)-dimensional real vector group \( \mathbb{R}^n \), \( \omega \) is a homeomorphism as shown in Example 2.4 below. As for \( \mathcal{A} \), it is not necessarily the direct product or the semi-direct product of \( \mathcal{R} \) and \( \mathcal{P} \), even if \( G \) is a connected Lie group, for both \( \mathcal{R} \) and \( \mathcal{P} \) are not necessarily normal subgroups of \( \mathcal{A} \). While \( \mathcal{A} \) is homeomorphic to the product space of \( \mathcal{R} \) and \( \mathcal{P} \), if \( G \) is locally compact. This fact is due to a remark by J. Keesling [6, Example V 6, p. 15]. In all statements above we can replace \( \mathcal{R} \) by \( \mathcal{P} \).

EXAMPLE 2.4. It is shown that \( \omega \) is a homeomorphism in the case where \( G \) is the \( n \)-dimensional real vector group \( \mathbb{R}^n \). Given arbitrary two positive numbers \( \delta_1 \) and \( \delta_2 \). Let \( V_1 \) be the set of all translations by \( b=(b_1,b_2,\ldots,b_n) \) such that \( |b_i|<\delta_2 \) \((i=1,2,\ldots,n)\), and let \( V_2 \) be the set of all regular linear transformations of \( \mathbb{R}^n \) determined by \( n \times n \) matrices \( a=(a_{ij}) \) such that \( |a_{ij}-\delta_{ij}|<\delta_3 \) \((i,j=1,2,\ldots,n)\) where \( \delta_{ij} \) denotes the Kronecker's delta. Then \( V_1 \) and \( V_2 \) are open neighborhoods of \( \mathcal{E} \) in \( \mathcal{R} \) and \( \mathcal{P} \) respectively. Take two positive numbers \( \sigma \) and \( \rho \) such that \( \sigma<\delta_1, \sigma/\rho<\delta_2 \). Let \( U \) be the open interval

\[
\{(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n \mid |x_i|<\sigma \text{ for all } i=1,2,\ldots,n\},
\]

let \( C \) be the compact interval

\[
\{(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n \mid |x_i|\leq\rho \text{ for all } i=1,2,\ldots,n\},
\]

and let

\[
W=\{u\in\mathcal{A} \mid u(x)-x\in U \text{ for all } x\in C\}.
\]

Then it is shown that \( W\subset V_1 V_2 \). In fact, express each \( u\in W \) in the from \( u=rua \), where let \( r \) be the translation by \( b=(b_1,b_2,\ldots,b_n) \), and \( a \) be the regular linear transformation determined by the matrix \( a=(a_{ij}) \). In the condition that \( u(x)-x\in U \) for all \( x\in C \), considering the case where \( x=o \), we have

\[
|b_i|<\sigma \quad (i=1,2,\ldots,n). \tag{1}
\]

Let \( j \) be any fixed number of \( 1,2,\ldots,n \). Considering the case where \( x=(0,\ldots,0,\rho,0,\ldots,0) \) \((\rho \text{ is the } j\text{-th component})\), we have

\[
-(\sigma+b_i)/\rho<a_{ij}-\delta_{ij}<(\sigma-b_i)/\rho \quad (i=1,2,\ldots,n).
\]

For \( x=(0,\ldots,0,-\rho,0,\ldots,0) \) \((-\rho \text{ is the } j\text{-th component})\), we have

\[
-(\sigma-b_i)/\rho<a_{ij}-\delta_{ij}<(a+b_i)/\rho \quad (i=1,2,\ldots,n).
\]

Thus

\[
\text{Max} \{-(\sigma+b_i)/\rho, -(\sigma-b_i)/\rho\} < a_{ij} - \delta_{ij} < \text{Min} \{(\sigma-b_i)/\rho, (\sigma+b_i)/\rho\}.
\]

And so

\[
|a_{ij}-\delta_{ij}|<\sigma/\rho \quad (i,j=1,2,\ldots,n) \tag{2}
\]

Relations (1) and (2) imply that \( u\in V_1 V_2 \). Consequently \( \omega^{-1} \) is continuous.
PROPOSITION 2.5. If $G$ is a connected Lie group, then $\mathcal{H}$ is a Lie group and
\[ \dim \mathcal{H} = \dim \mathcal{K} + \dim \mathcal{A} < +\infty. \]

PROOF. $\mathcal{H}$ is separable metrizable. 
In fact since $G$ is connected locally compact metrizable, $G$ is separable. Hence (1) follows from a remark of J. Keesling \[6, \text{Remark III. 4, pp. 10, 11} \] that if $X$ is a separable metric locally compact space and $Y$ is a separable metric space then the family of all continuous mappings of $X$ into $Y$ is separable metrizable. Therefore the covering dimension coincides with the inductive dimension for $\mathcal{H}$. On the other hand D. H. Lee and T. -S. Wu \[8\] proved that the automorphism group of a finite-dimensional connected locally compact group is a Lie group under the topology usually used for automorphism groups, which coincides with the compact-open topology under further supposition of local connectedness on the base group. Therefore $\mathcal{A}$ is a Lie group under the compact-open topology. Let $\mathcal{K} \times \mathcal{A}$ be the direct product group of topological groups $\mathcal{K}$ and $\mathcal{A}$. Since $\mathcal{K}$ is inverse-isomorphic to $G$ and $\mathcal{A}$ is a Lie group, considering (1), $\mathcal{K} \times \mathcal{A}$ is a Lie group satisfying the 2nd axiom of countability. Thus $\mathcal{K} \times \mathcal{A}$ is covered by a countable number of compact neighborhoods $W_i$ ($i=1,2, \ldots$). Let $\omega$ be the mapping $(r,u) \rightarrow ru$ of $\mathcal{K} \times \mathcal{A}$ onto $\mathcal{N} = \mathcal{K} \mathcal{A}$, then $\omega$ is a continuous bijection and
\[ \mathcal{N} \text{ is covered by a countable number of compact sets } \omega(W_i)(i=1,2, \ldots). \]
Since $W_i$ is homeomorphic to $\omega(W_i)$,
\[ \dim W_i = \dim \omega(W_i). \]
And since $\mathcal{K} \times \mathcal{A}$ is homogeneous,
\[ \dim W_i = \dim (\mathcal{K} \times \mathcal{A}) (i=1,2, \ldots) \]
By (2), (3), (4), and sum theorem of dimension we have
\[ \dim \mathcal{N} \leq \dim (\mathcal{K} \times \mathcal{A}). \]
It is evident that
\[ \dim (\mathcal{K} \times \mathcal{A}) = \dim \omega(W_i) \leq \dim \mathcal{N}. \]
Thus by (5) and (6), we have
\[ \dim \mathcal{N} = \dim (\mathcal{K} \times \mathcal{A}). \]
Since $\mathcal{K}$ and $\mathcal{A}$ are Lie groups, we have
\[ \dim (\mathcal{K} \times \mathcal{A}) = \dim \mathcal{K} + \dim \mathcal{A}. \]

§ 3. Construction of a closed subgroup homeomorphic to Hilbert space.

It is known that a locally compact metrizable topological group $G$ which is not totally disconnected admits a nontrivial flow \[6, \text{Cor. V. 5, p. 15} \], so that the full homeomorphism group $\mathcal{H}$ of $G$ contains a subspace homeomorphic to an infinite-dimensional Hilbert space \[6, \text{Th. III. 2, p. 6} \]. If $G$ is a connected Lie group, such a subspace is not contained in $\mathcal{J}$ by Proposition 2.5. It seems to us that such a subspace must be found in $\mathcal{J}$ and intersects each coset of $\mathcal{A}$ at most one point. Indeed we can construct a closed subgroup of $\mathcal{J}$ of the character in this section in the case where $G$ is a finite-dimensional connected locally compact nontrivial topological
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Let $G$ be a topological group, $A$ its subset having the property (P) (see Definition 1.2), and $B$ a subset of $G$ which contains the identity $e$ of $G$. Let $U=AB$. Suppose that $U$ is locally compact and each $x\in U$ can be written in the form $x=ab (a\in A, b\in B)$ uniquely and continuously. Moreover let $\mathcal{G}$ be a group of homeomorphisms of $A$ onto itself, and for each $f\in \mathcal{G}$ let $f^*$ be the mapping of $U$ onto itself such that $f^*(x)=f(a)'b$ ($x=ab \in U, a\in A, b\in B$). Let $\mathcal{G}^*$ be the set of all $f^*$ associated with $f\in \mathcal{G}$ as above, and $\omega$ be the associating mapping. Then $\mathcal{G}$ and $\mathcal{G}^*$ with compact-open topologies are topological transformation groups of $A$ and $U$ respectively, and isomorphic as topological groups by the mapping $\omega$. And each $f\in \mathcal{G}$ is the restriction of $f^*$ on $A$.

Moreover we will use the following two structure theorems.

**Theorem of K. Iwasawa** ([5], Th. 13, p. 549; [11], Th. 5, p. 364). A connected locally compact group $G$ has maximal compact subgroups, and all such subgroups are connected and are conjugate to each other. Let $K$ denote one of them. Then $G$ contains subgroups $H_1, H_2, \ldots, H_s$ all isomorphic to the 1-dimensional vector group and such that any element $g\in G$ can be decomposed uniquely and continuously in the form

$$g=kh_1h_2\cdots h_s,$$

where $k\in K$, $h_i\in H_i$.

In particular, the space of $G$ is the product of the compact space of $K$ and that of $H_1\times H_2\times \cdots \times H_s$, which is homeomorphic to the $s$-dimensional Euclidean space.

In connection with the compact subgroup $K$ in this theorem we will use the following

**Theorem of J. von Neumann** ([10], p. 335). Let $K$ be a compact group of finite dimension $r$. Then $K$ contains a local Lie subgroup $L$ homeomorphic to an $r$-dimensional open cube, and a 0-dimensional compact normal subgroup $Z$ such that i) $LZ$ is an open neighborhood of the identity in $K$, and ii) every element $k\in LZ$ can be decomposed uniquely and continuously in the form

$$k=1z,$$

where $l\in L$, $z\in Z$.

We can take a subset $B$ of $L$, homeomorphic to an $r$-dimensional closed cube and the center of the cube corresponds to the identity by the homeomorphism. Let $U=BJ$. Hereafter we will use these notations and those in the above two theorems, and let $G$ be a finite-dimensional connected locally compact nontrivial topological group throughout the rest of this section. Then either $r$ or $s$ in the above theorems must be positive. We consider first the case where $r$ is positive.

1°. Let $B^*$ be the closed unit ball in Cartesian $r$-space, $\mathcal{G}$ be the homeomorphism group of $B^*$ constructed as in Lemma 1.1 and $\mathcal{G}_0$ be the homeomorphism group of $B$ naturally induced from $\mathcal{G}$ by a certain homeomorphism of $B^*$ onto $B$.

2°. For each $u_0\in \mathcal{G}_0$ let $u_1$ be the mapping of $U$ onto itself defined as follows :

$$u_1(x)=u_0(b)'z,$$

where $x=bz\in U$, $b\in B$, and $z\in Z$.

Let $\mathcal{G}_1$ be the set of all $u_1$ associated with $u_0\in \mathcal{G}_0$. Each $u_1\in \mathcal{G}_1$ leaves every boundary point of $U$ in $K$ fixed. In fact, let $B^0$ be the interior of $B$ in $L$, then for any $x\in U$—
BoZ such that \( x = bz (b \in B, z \in Z) \), we have \( b \in B - B_0 \). And so \( u_1(x) = u_0(b) \cdot z = bz = x \), i.e. \( u_1 \) leaves every point of \( U - BoZ \) fixed. Since \( B''OZ \subseteq \) (the interior of \( U \) in \( K \)), \( u_1 \) leaves every boundary point of \( U \) in \( K \).

3°. For each \( u_1 \in \mathcal{Q}_1 \), let \( u_2 \) be the mapping of \( K \) onto itself defined as follows :
\[
u_2(x) = u_1(x) \quad \text{if} \quad x \in U \quad \text{and} \quad u_2(x) = x \quad \text{if} \quad x \in K - U.
\]
Let \( \mathcal{Q}_2 \) be the set of all \( u_2 \) associated with \( u_1 \in \mathcal{Q}_1 \).

4°. Let \( H = H_1H_2 \cdots H_s \). \( H \) is homeomorphic to \( s \)-dimensional Euclidean space, though it is not necessarily a subgroup of \( G \). For each \( u_2 \in \mathcal{Q}_2 \), let \( u \) be the mapping of \( G \) onto itself defined as follows :
\[
u(x) = u_2(k) \cdot h, \quad \text{where} \quad x = kh \in G, k \in K, \quad \text{and} \quad h \in H.
\]
Let \( \mathcal{Q} \) be the set of all \( u \) associated with \( u_2 \in \mathcal{Q}_2 \).

By Lemmas 1.1, 1.3 and 3.1 the set \( \mathcal{Q} \) thus constructed becomes a topological transformation group acting on \( G \) under the compact-open topology, isomorphic to the group \( \mathcal{G}^+ (I) \) of all orientation preserving homeomorphisms of the closed interval \([0,1]\) onto itself, and each \( u \in \mathcal{Q} \) leaves the identity \( e \) of \( G \) fixed. Hence \( \mathcal{Q} \) is a subgroup of \( \mathcal{G} \), homeomorphic to \( l_s \).

Next to prove \( \mathcal{Q} \cap \mathcal{A} = \{ e \} \), we prepare the following

**Lemma 3.2.** Let \( G \) be a connected topological group. If \( u \) is an automorphism of \( G \) which is somewhere the identity, that is \( u | V \) is identity for some nonempty open subset \( V \) of \( G \), then \( u \) is the identity mapping of \( G \).

**Proof.** Take a point \( x_0 \in V \) and then take a neighborhood \( V_1 \) of \( e \) in \( G \) such that \( V_1x_0 \subseteq V \). For any point \( x \) of \( V_1 \), we have
\[
x_0x = u(x_0x) = u(x) \cdot x_0, \quad \text{and so} \quad u(x) = x.
\]
Hence \( u \) leaves every point of \( V_1 \) fixed. Therefore \( u \) leaves every point of \( G \) fixed, since \( G \) is a connected topological group.

Now let \( u \) be any element of \( \mathcal{Q} \cap \mathcal{A} \). Since \( u \) leaves each point of the nonempty open subset \( (K - U)H \) of \( G \) fixed, \( u \) must be the identity mapping \( e \) of \( G \) by Lemma 3.2. Therefore \( \mathcal{Q} \cap \mathcal{A} = \{ e \} \). Hence \( \mathcal{Q} \) intersects each coset of \( \mathcal{A} \) at most one point.

Secondary we can treat the case where \( s \) is positive as well. Let \( B' \) be a neighborhood of \( e \) in \( H \) which is homeomorphic to an \( s \)-dimensional closed cube, and \( \mathcal{Q}_0 \) be the homeomorphism group of \( B' \) constructed similarly to the first step 1° in the case where \( r \) is positive. For each \( u_0 \in \mathcal{Q}_0 \), let \( u_1 \) be the mapping defined as follows :
\[
u_1(x) = u_0(x) \quad \text{if} \quad x \in B', \quad \text{and} \quad u_1(x) = x \quad \text{if} \quad x \in H - B'.
\]
Let \( \mathcal{Q}_1 \) be the set of all \( u_1 \) associated with \( u_0 \in \mathcal{Q}_0 \). For each \( u_1 \in \mathcal{Q}_1 \), let \( u \) be the mapping defined as follows :
\[
u(x) = k \cdot u_1(h), \quad \text{where} \quad x = kh \in G, k \in K, \quad \text{and} \quad h \in H.
\]
Let \( \mathcal{Q} \) be the set of all \( u \) associated with \( u_1 \in \mathcal{Q}_1 \). Then \( \mathcal{Q} \) is a subgroup of \( \mathcal{G} \) isomorphic to \( \mathcal{G}^+ (I) \), and intersects each coset of \( \mathcal{A} \) at most one point.

Now we can show that the group \( \mathcal{Q} \) is closed in \( \mathcal{G} \) in the case where \( r \) or \( s \) is positive.

We consider the case where \( r \) is positive. Let \( \pi_K \) be the natural projection of \( G \) onto \( K \), \( \pi_B \) be the projection of \( U \) onto \( B \), and \( \pi \) be the projection \( x \mapsto \| x \| \) of \( B \) onto
I, where the norm \( \| \| \) is the one naturally induced from that of the closed unit ball \( B^* \) in Cartesian \( r \)-space. Let a net \((u^n)\) in \( \mathcal{Q} \) converges to a \( v \in \mathcal{H} \), and let each \( u^n \) be defined by \( f^n, u^n, u^n, u^n(f^n \in \mathcal{H}^*(I), u^n \in \mathcal{Q}) \) successively by the way of the four steps \( 1^o-4^o \) stated before. Then
\[
f^n \pi = u^n \text{ on } B, u^n \pi_B = \pi_B u^n \text{ on } U, u^n \mid U = u^n, \text{ and } u^n \pi_K = \pi_K u^n \text{ on } G.
\]
By the similar method as in the proof of Theorem I.5, we have successively
\[
(1) u^n \pi_K \text{ in } \mathcal{G}(K;K),
(2) u^n \pi_U \text{ in } \mathcal{G}(U;U),
(3) u^n \pi_B \text{ in } \mathcal{G}(B;B),
(4) f^n \pi \pi_K \pi_K \text{ in } \mathcal{G}(I;I).
\]
Define the mapping \( f^* \) of \( I \) into itself as follows:
\[
f^*(t) = \pi_B \pi_K \pi_K(t) (t \in I).
\]
Then the mapping \( f^* \) is an orientation preserving homeomorphism of \( I \) onto itself. Starting from \( f^* \in \mathcal{H}^*(I) \), define mappings \( u_0^*, u_1^*, u_2^*, \) and \( u^* \) by way of the four steps \( 1^o-4^o \) stated above. Then by (1), (2), (3), and (4) we have the following relations successively:
\[
u_0^* = \pi_B \pi_K \pi_K | B, u_1^* = \pi_K | U, u_2^* = \pi_K \pi_K | K, u^* = v.
\]
Therefore \( v \) is an element of \( \mathcal{Q} \). This shows that \( \mathcal{Q} \) is closed in \( \mathcal{H} \), and so in \( \mathcal{G} \) also.

We can treat the case where \( s \) is positive as well more simply.

From the results obtained we have the following

**Theorem 3.3** If \( G \) is a finite-dimensional connected locally compact nontrivial topological group, then there exists a closed subgroup \( \mathcal{Q} \) of \( \mathcal{G} \), which intersects each left (or right) coset of \( \mathcal{G} \) at most one point and is isomorphic to \( \mathcal{H}^*(I) \).

**Corollary 3.4** If \( G \) is a finite-dimensional connected locally compact nontrivial group, then the full homeomorphism group \( \mathcal{H} \) of \( G \) is not locally compact, not closed in the space \( \mathcal{G}(G) \) of all continuous mappings of \( G \) into itself, not complete with respect to the uniformity of compact convergence defined by the right uniformity of \( G \).

**References**

