Local Compactness for Families of Continuous Mappings and Homeomorphism Groups

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Abstract

Let $X$ be a topological space, $Y$ a uniform space, $\mathcal{C}(X;Y)$ the family of all continuous mappings of $X$ into $Y$ with the compact-open topology, and $\mathcal{C}$ a subspace of $\mathcal{C}(X;Y)$.

**Theorem.** Consider two conditions: (a) $\mathcal{C}$ is locally closed in $\mathcal{C}(X;Y)$ and locally equicontinuous, and (b) $\mathcal{C}$ is locally compact. Then (a) implies (b) in each case of the following (1), (2), and (3): (1) $X$ contains but a finite number of components and $Y$ is locally compact, complete, (2) $X$ is compact and $Y$ is locally compact, (3) $Y$ is compact. And (b) implies (a), if $X$ is locally compact and $Y$ is Hausdorff.

Using this, conditions for homeomorphism groups to become locally compact transformation groups are obtained.

Introduction. General theory of topological transformation groups has been developed for locally compact transformation groups in relation to the conjecture of Hilbert and Smith (cf. e.g., D. Montgomery and L. Zippin [8]). And so, for the problem when a homeomorphism group becomes a Lie transformation group, to make the group locally compact under a suitable topology is the first gate for us to go through. The purpose of this paper is to show the conditions for families of continuous mappings between two spaces to become locally compact under the compact-open topology (Theorem 1), and to apply the results to the problem when homeomorphism groups become locally compact transformation groups (Theorem 2). The problem for compact case was treated by S. B. Myers [9] and his results were generalized by J. Dieudonné [4], who gave sufficient conditions for a homeomorphism group of a space to be embedded in a locally compact transformation group of the
space as a dense subgroup. This problem of J. Dieudonné, as he said, offers difficulties as regards the way in which the group must be completed so as to become locally compact. Noting the importance of local closedness and local equicontinuity for the problem, the difficulties can be avoided in the present paper.

On account of the assumption that the base space is complete, our main theorems do not imply the theorem of D. van Dantzig and B. L. van der Waerden [3] on isometry groups, and so we will newly give a generalization of it (Proposition 1). In §3, sufficient conditions for families of homeomorphisms to be closed in the space of all continuous mappings will be given (Propositions 2, 3, and 4), then the relation between our results and theorems of R. Arens, J. Dieudonné, and D. van Dantzig and B. L. van der Waerden [1, Th. 7, p. 604; 4, Prop. 12, p. 675; 3, Satz I, p. 370, resp.] will be clear.

1. Local compactness for families of continuous mappings.

Under the compact-open topology many valuable results have been obtained. And a set-entourage uniformity on the family of all continuous mappings of a euclid-like uniform space into itself must be the uniformity of compact convergence, if both joint-continuity and continuity of mapping composition are required [7, Th. 6 and Th. 7, p. 289]. Thus we will set importance on the compact-open topology.

**Lemma 1 (Ascoli-Bourbaki).** Let \( X \) be a topological space, \( Y \) a uniformizable space, \( C(X;Y) \) the family of all continuous mappings of \( X \) into \( Y \), and \( C \) a subfamily of \( C(X;Y) \). Consider the following three conditions:

1. \( C \) is equicontinuous for at least one compatible uniformity on \( Y \), and \( C(x) \) is relatively compact for each \( x \in X \).
2. \( C \) is equicontinuous for every compatible uniformity on \( Y \), and \( C(x) \) is relatively compact for each \( x \in X \).
3. \( C \) is relatively compact in \( C(X;Y) \) under the compact-open topology.

Then (1) implies (3), and (2) implies (1). If \( X \) is locally compact, (3) implies (2).

When \( Y \) is assumed to be Hausdorff moreover, Lemma 1 is well-known (cf. N. Bourbaki [2, Cor. 3, p. 292], H. Schubert [10, Satz 3, p. 149], etc.). While if we note that the closure of a compact subset of a uniformizable space is compact, we can do without the assumption.

**Remark.** (1) implies that \( C \) is relatively compact in \( C(X;Y) \) under any topology that is finer than the point-open topology and is coarser than the compact-open topology. If \( C \) is relatively compact in \( C(X;Y) \) under a jointly continuous topology on \( C(X;Y) \), then (2) holds without the assumption that \( X \) is locally compact.
LEMMA 2. Let $X$ be a connected topological space, and $Y$ a locally compact, complete uniform space. If $\mathcal{G}$ is an equicontinuous family of mappings of $X$ into $Y$, then the following (1) implies (2):

1. $\mathcal{G}(x)$ is relatively compact for at least one $x \in X$.
2. $\mathcal{G}(x)$ is relatively compact for each $x \in X$.

This is well-known [9, Th. 4.1, p. 497; 4, pp. 676, 677]. Let $E$ be the set of all points $x$ in $X$ such that $\mathcal{G}(x)$ has compact closure. If we note that $\text{Cl}[\mathcal{G}(x)]$ is precompact for each point $x$ of $\text{Cl}E$, then the proof will be easier.

REMARK. A generalization: If $X$ consists of $n(<\infty)$ connected components, choose points $x_i (i=1,2,\ldots,n)$ one by one from each component, and replace (1) by

1'. $\mathcal{G}(x_i)$ is relatively compact for $i=1,2,\ldots,n$. Then (2) holds.

DEFINITION. $\mathcal{G}$ is locally equicontinuous at $u \in \mathcal{G}$ if there exists an equicontinuous neighborhood of $u$ in $\mathcal{G}$.

S. B. Myers [9, Footnote 5, p. 499] has remarked that if $X$ is a locally compact or first countable, and connected space and $Y$ is a locally compact, complete, metric space, then under the compact-open topology local equicontinuity of $\mathcal{G}$ is equivalent to local compactness of $\mathcal{G}$. It is valid if we understand "local compactness" as such that each $u \in \mathcal{G}$ has a compact neighborhood in the space $\mathcal{C}(X;Y)$. If we understand it as such "local compactness of $\mathcal{G}$ in itself" the assertion is false, for the additive group of rational numbers acting on the real line as translations is an counter-example. Moreover we can show that "closedness of $\mathcal{G}$ in $\mathcal{C}(X;Y)$ and local equicontinuity of $\mathcal{G}$ "implies" local compactness of $\mathcal{G}$ in itself", but the converse is not true, as we may only consider the general linear group on cartesian $n$-space.

Our main theorem is the following.

THEOREM 1. Let $X$ be a topological space, $Y$ a uniform space, $\mathcal{C}(X;Y)$ the family of all continuous mappings of $X$ into $Y$, and $\mathcal{S}$ a subfamily of $\mathcal{C}(X;Y)$. Consider the following two conditions under the compact-open topology:

(a) $\mathcal{S}$ is locally closed at $u \in \mathcal{S}$ in $\mathcal{C}(X;Y)$, and locally equicontinuous at $u$.

(b) $\mathcal{S}$ has a compact neighborhood of $u$ in $\mathcal{S}$.

Then (a) implies (b) in each case of the following, (1), (2), and (3):

1. $X$ contains but a finite number of connected components, and $Y$ is locally compact, complete.

2. $X$ is compact, and $Y$ is locally compact.

3. $Y$ is compact.

And (b) implies (a) if $X$ is locally compact and $Y$ is Hausdorff.

To prove Theorem 1 we need the following.

LEMMA 3. If $\mathcal{S}$ is locally equicontinuous at $u \in \mathcal{S}$, then in each case of
(1), (2), and (3) in Theorem 1, there exists such an equicontinuous neighborhood of $u$ in $\mathfrak{C}$ that is relatively compact in $\mathcal{C}(X;Y)$.

**Proof.** Case (1): Let $W$ be an equicontinuous neighborhood of $u$ in $\mathfrak{C}$. Choose points $\{a_i\}$ one by one from each component of $X$, and let $V_i$ be a compact neighborhood of $u(a_i)$. By the joint-continuity on $W \times X$, there exist a neighborhood $W_i$ of $u$ in $\mathfrak{C}$ and a neighborhood $U_i$ of $a_i$ such that $W_i(U_i) \subseteq V_i$ and $W_i \subseteq W$. Let $W^* = \cap W_i$. Then $W^*(a_i)$ is relatively compact for each $i$. Hence by Lemma 1 and Remark to Lemma 2, $W^*$ is relatively compact in $\mathcal{C}(X;Y)$.

Case (2): For each $x \in X$, take a compact neighborhood $V_x$ of $u(x)$, an equicontinuous neighborhood $W_x$ of $u$ in $\mathfrak{C}$, and an open neighborhood $U_x$ of $x$ such that $W_x(U_x) \subseteq V_x$. From the open covering $\{U_x|x \in X\}$ of $X$ choose a finite subcovering. Then the intersection of the corresponding $W_x$ is the desired neighborhood of $u$ in $\mathfrak{C}$.

Case (3): For every equicontinuous neighborhood $W$ of $u$ in $\mathfrak{C}$, $W(x)$ is relatively compact for each $x \in X$. Thus $W$ is relatively compact in $\mathcal{C}(X;Y)$ by Lemma 1.

**Proof of Theorem 1.** (a) implies (b): Since $\mathfrak{C}$ is locally closed at $u$ in $\mathcal{C}(X;Y)$, there exists a neighborhood $W_1$ of $u$ in $\mathcal{C}(X;Y)$ such that $\mathfrak{C} \cap W_1$ is the intersection of $W_1$ and a closed subset of $\mathcal{C}(X;Y)$. Take a neighborhood $W_x$ of $u$ in $\mathcal{C}(X;Y)$ such that $\text{Cl} W_x \subseteq W_1$. By Lemma 3 there exists an equicontinuous neighborhood $W$ of $u$ in $\mathfrak{C}$ such that $W \subseteq W_x \cap \mathfrak{C}$ and $W$ is relatively compact in $\mathcal{C}(X;Y)$. Then it is easy to see that the closure of $W$ in $\mathcal{C}(X;Y)$ is contained in $\mathfrak{C}$.

(b) implies (a): Since $\mathcal{C}(X;Y)$ is Hausdorff and $\mathfrak{C}$ has a compact neighborhood of $u$ in $\mathfrak{C}$, $\mathfrak{C}$ is locally closed at $u$ in $\mathcal{C}(X;Y)$. Moreover a compact neighborhood $W$ of $u$ in $\mathfrak{C}$ is a closed and compact subset of $\mathcal{C}(X;Y)$ also. Therefore by Lemma 1, $W$ is equicontinuous.

### 2. Local Compactness for Homeomorphism Groups.

We will apply Theorem 1 to the problem under what circumstances homeomorphism groups become locally compact topological transformation groups.

**Lemma 4** (R. Ellis [5, Th., p.373]). Let $\mathfrak{G}$ be a group with a locally compact Hausdorff topology such that multiplication is continuous. Then $\mathfrak{G}$ is a topological group.

**Theorem 2.** Let $X$ be a Hausdorff uniform space, $\mathcal{C}(X)$ the family of all continuous mapping of $X$ into itself, and $\mathfrak{G}$ a group of homeomorphisms of $X$ onto itself. Consider the following two conditions under the compact-open topology:

(a) $\mathfrak{G}$ is locally closed at the identity $e$ in $\mathcal{C}(X)$, and has an equicontinuous
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neighborhood of \( e \) in \( \mathcal{G} \).

(b) \( \mathcal{G} \) is a locally compact transformation group acting on \( X \).

Then (a) implies (b) if \( X \) is compact or if \( X \) is locally compact, complete, and contains but a finite number of connected components. And (b) implies (a) if \( X \) is locally compact.

**Proof.** With consideration for Theorem 1, we have only to prove that (a) implies that \( \mathcal{G} \) is a topological transformation group of \( X \) under the compact-open topology \( \tau \). The joint-continuity and the continuity of multiplication under \( \tau \) follow from the fact that \( \mathcal{G} \) is a family of continuous mappings and \( X \) is a locally compact uniform space [4, Lemma 4, p. 661; Prop. 3, p. 662].

By Theorem 1, \( \mathcal{G} \) is locally compact under \( \tau \). As \( X \) is Hausdorff, \( \mathcal{G} \) is so. Consequently to verify the continuity of inverse operation, we have only to apply Lemma 4.

**Proposition 1.** Let \( X \) be a Hausdorff uniform space that is locally compact and contains but a finite number of connected components, or compact. Let \( \mathcal{G} \) be a group of homeomorphisms of \( X \) onto itself. If under the compact-open topology,

1. \( \mathcal{G} \) has a uniformly equicontinuous, symmetric neighborhood of the identity in \( \mathcal{G} \), and
2. \( \mathcal{G} \) is locally closed at the identity in \( \mathcal{G}(X) \),

then \( \mathcal{G} \) is a locally compact transformation group of \( X \).

To prove Proposition 1 we prepare the following two lemmas.

**Lemma 5.** Let \( X \) be a locally compact uniform space, and \( \mathcal{G} \) a group of homeomorphisms of \( X \) onto itself. Under the compact-open topology, if \( \mathcal{G} \) has an equicontinuous symmetric neighborhood of the identity in \( \mathcal{G} \), then \( \mathcal{G} \) is a topological transformation group acting on \( X \).

**Proof.** The joint-continuity and the continuity of multiplication in \( \mathcal{G} \) follow easily. The continuity of inverse operation in \( \mathcal{G} \) is shown as follows. Let \( W \) be an equicontinuous symmetric neighborhood of the identity \( e \) in \( \mathcal{G} \), then if a net \( \{u_i\} \) in \( W \) converges to \( e \), \( \{u_i^{-1}\} \) converges to \( e \). Thus for any neighborhood \( W' \) of \( e \), \( W'^{-1} \) is a neighborhood of \( e \) also, and so \( \mathcal{G} \) has arbitrarily small symmetric neighborhood of \( e \) in \( \mathcal{G} \). Hence from the fact that in a semitopological group (in the sense of T. Husain [6]) \( \{W, u\} \) is a fundamental system of neighborhoods of \( u \subseteq \mathcal{G} \) for a fundamental system \( \{W_i\} \) of neighborhoods of \( e \), the inverse operation in \( \mathcal{G} \) is continuous.

**Lemma 6.** Let \( X \) and \( \mathcal{G} \) be the same as in Proposition 1. If \( \mathcal{G} \) has a uniformly equicontinuous symmetric neighborhood of the identity in \( \mathcal{G} \), then \( \mathcal{G} \) has an equicontinuous symmetric neighborhood \( W \) of the identity that is relatively compact in \( \mathcal{G}(X) \).

**Proof.** The case where \( X \) is compact is evident. Choose points \( \{x_i\} \) one by one from each component of \( X \). Take a compact neighborhood \( U_i \) of \( x_i \),
then there exists a neighborhood $W_i$ of $e$ in $O$ such that 
$Cl[\{W_i(x_i)\}] \subseteq U_i \quad (i=1,2,\ldots,n)$.

By Lemma 5, there exists a uniformly equicontinuous, symmetric neighborhood $W$ of $e$ in $O$ such that $W \subseteq \bigcap W_i$. Then $Cl[\{W(x_i)\}]$ is compact for each $i$. Let $E$ be the set of all points $x$ such that $Cl[\{W(x)\}]$ is compact. $E$ is open and intersects each component of $X$. The closedness of $E$ can be shown by the same method as in [4, p. 677]. Consequently $E$ coincides with $X$. Hence by Lemma 1, $ClW$ in $O(X)$ is compact.

Proof of Proposition 1. Since $O$ is locally closed at $e$ in $O(X)$, we can take such a $W$ in Lemma 6 that the closure of $W$ in $O(X)$ is contained in $O$. Then $W$ is a compact neighborhood of $e$ in $O$.

Remark. Let $O(X)$ be the family of all permutations continuous on every compact subset of $X$. The set $W$ in Lemma 6 is a symmetric neighborhood of $e$ which is relatively compact in $O(X)$ under the topology of compact convergence also (cf. Propositions 2 and 4 in the next section). Thus the assertion for the case b) of Proposition 12 in [4] follows. And the assertion for the case a) of it follows similarly (cf. Lemma 3 in the preceding section and Proposition 3 in the next section).

3. Limit of a net of homeomorphisms.

When the limit of a net of self-homeomorphisms is a self-homeomorphism also? It was treated by S. B. Myers [9, Lemma 5.1, p. 500], whose result corresponds to the metric case for our Proposition 5 below, and is used to establish the relation between compactness and equicontinuity for homeomorphism groups. As it is interesting for us, we will generalize his result.

Let $X$ be a Hausdorff uniform space, $O(X)$ the family of all continuous mappings of $X$ into itself with a jointly continuous topology $\tau$, and $\{u_i\}_{\lambda \in \Lambda}$ a net of homeomorphisms of $X$ onto itself. Assume that $u_i$ converges to $u \in O(X)$ and $\{u_i^{-1}\}$ is equicontinuous. These notations and assumption will be kept throughout this section.

Lemma 7. $u_i^{-1}[u(x)]$ converges to $x$ for each $x \in X$.

Proof. Use the equicontinuity of $\{u_i^{-1}\}$ at $u(x)$ and the joint-continuity of $\tau$.

Lemma 8. $u$ is a homeomorphism of $X$ into itself.

Proof. If $u(x_i)=u(x_2)$ $(x_1, x_2 \in X)$, then by the equicontinuity of $\{u_i^{-1}\}$ at $u(x_i)$ and the joint-continuity of $\tau$, $(x_1, x_2)$ belongs to any entourage of $X$. Thus $u$ is an injection. Next let $u^{-1}(x^*)=x$ $(x^* \in u(X))$. By the equicontinuity of $\{u_i^{-1}\}$ at $x^*$ and Lemma 7, for any entourage $\mathcal{B}$ of $X$ there exists a neighborhood $U$ of $x^*$ such that $u^{-1}(U \cap u(X)) \subseteq \mathcal{B}(x)$. Thus $u^{-1}$ is continuous.
LEMMA 9. If $X$ is locally compact, then $u(X)$ is open.

**Proof.** For any point $p$ of $u(X)$ let $q = u^{-1}(p)$. Take entourages $\mathcal{B}$ and $\mathcal{U}$ such that $\mathcal{B}(q)$ is compact and $\mathcal{U} \subseteq \mathcal{B}$. Since $\{u_i^{-1}\}$ is equicontinuous at $p$, there exists a neighborhood $U$ of $p$ such that

$$(1) \quad (u_i^{-1}(p), u_i^{-1}(x)) \in \mathcal{U} \text{ for any } x \in U \text{ and any } u_i.$$  

Then $U \subseteq u(X)$ as shown below. Let $x$ be any point of $U$. Since $u_i^{-1}u(q)$ converges to $q$, there is a $\lambda_0 \in \Lambda$ such that

$$(2) \quad (q, u_i^{-1}(p)) \in \mathcal{U} \text{ for any } \lambda > \lambda_0.$$  

By $(1)$ and $(2)$, we have

$$u_i^{-1}(x) \in \mathcal{B}(q) \text{ for any } \lambda > \lambda_0.$$  

Thus there exists a subnet $\{\lambda\}$ of $\{\lambda\}$ such that $u_i^{-1}(x)$ converges in $\mathcal{B}(q)$.

Let $y$ be the limit, then $u(y) = x$.

LEMMA 10. If $X$ is locally compact and $\{u_i\}$ is uniformly equicontinuous, then $u(X)$ is closed.

**Proof.** For any point $y$ of $\text{Cl}[u(X)]$ take an entourage $\mathcal{U}$ such that $\mathcal{U}(y)$ is compact. There exists a net $\{y_\alpha\}$ in $u(X) \cap \mathcal{U}(y)$ which converges to $y$. Let $x_\alpha = u^{-1}(y_\alpha)$, then $\{x_\alpha\}$ is a Cauchy net. In fact, for any entourage $\mathcal{B}$ take an entourage $\mathcal{U}$, such that $\text{Cl}[\mathcal{B} \cap \mathcal{U}] \subseteq \mathcal{B}$. Since $\{u_i^{-1}\}$ is equicontinuous at $y$, there exists a neighborhood $U$ of $y$ such that

$$(u_i^{-1}(y), u_i^{-1}(y')) \in \mathcal{B}, \text{ for any } \lambda \in \Lambda \text{ and any } y' \in U.$$  

On the other hand $y_\alpha$ converges to $y$, and so there is a $\mu_0$ such that $u(x_\alpha) \in U$ for any $\mu \geq \mu_0$. Hence

$$(u_i^{-1}u(x_\alpha), u_i^{-1}u(x_\beta)) \in \mathcal{B}, \text{ for any } \lambda \text{ and any } \mu, \nu \geq \mu_0.$$  

Therefore $\{x_\alpha, x\} \in \mathcal{B}$ for any $\mu, \nu \geq \mu_0$ by Lemma 5.

Now take an entourage $\mathcal{U}$, such that $\mathcal{U} \cup \mathcal{U} \subseteq \mathcal{U}$. Since $\{u_i\}$ is uniformly equicontinuous, there exists an entourage $\mathcal{U}_2$ such that

$$(u_i^{-1}(y), u_i^{-1}(y')) \in \mathcal{U}_2, \text{ for any } \lambda \in \Lambda \text{ and any } y' \in U.$$  

Thus there is a $\mu_1$ such that

$$(1) \quad (u_i(x_\alpha), u_i(x_\beta)) \in \mathcal{U}_2, \text{ for any } \lambda \text{ and any } \mu, \nu \geq \mu_1.$$  

Take an entourage $\mathcal{U}_2$ such that $\mathcal{U}_2 \cap \mathcal{U}_2 \subseteq \mathcal{U}_1$. There is a $\mu_2$ such that

$$(2) \quad (y, u(x_\alpha)) \in \mathcal{U}_2, \text{ for any } \mu \geq \mu_2.$$  

Fix any $\omega$ such that $\omega > \mu_1, \mu_2$. Then

$$(3) \quad (u(x_\alpha), u(x_\beta)) \in \mathcal{U}_1, \text{ for any } \mu > \mu_2.$$  

As $u_i(x_\alpha)$ converges to $u(x_\alpha)$, there is a $\lambda_1$, such that

$$(4) \quad (u(x_\alpha), u_i(x_\alpha)) \in \mathcal{U}_1, \text{ for any } \lambda > \lambda_1.$$  

By $(1)$, $(2)$, $(3)$, and $(4)$, we have

$$(y, u(x_\alpha)) \in \mathcal{U}_1 \text{ for any } \lambda > \lambda_1 \text{ and any } \nu > \omega.$$  

Since $\mathcal{U}(y)$ is compact, for any fixed $\epsilon > \lambda_1$, there exists a subnet $\{x_\beta\}$ of $\{x_\alpha | \nu > \omega\}$ such that $u_i(x_\beta)$ converges and so $x_\beta$ converges. Let $x$ be the limit.

1) A method used in an unpublished paper by R. Tahata on the pointwise limit of a sequence of onto-isometries is used here in a more generalized form.
of \( \{ x_\alpha \} \), then \( y = u(x) \).

**Lemma 11.** If \( X \) is complete, then \( u(X) \) is closed.

**Proof.** In the proof of Lemma 10, \( \{ x_\alpha \} \) is a Cauchy net. Hence \( \{ x_\alpha \} \) converges. Let \( x \) be the limit, then \( y = u(x) \).

**Proposition 2.** If \( X \) is locally compact and contains but a finite number of connected components, and \( \{ u_\alpha \} \) is uniformly equicontinuous, then \( u \) is a homeomorphism of \( X \) onto itself.

**Proof.** The set \( u(X) \) is open and closed by Lemmas 9 and 10. Hence each component of \( X \) is contained in \( u(X) \) or does not intersect \( u(X) \), and each component of \( u(X) \) is a component of \( X \) also. Since \( u \) is a homeomorphism of \( X \) onto \( u(X) \), the number of components of \( X \) equals that of \( u(X) \). Thus \( u(X) = X \).

**Proposition 3.** If \( X \) is locally compact, complete, and contains but a finite number of components, then \( u \) is a homeomorphism of \( X \) onto itself.

**Proof.** This follows from Lemmas 9 and 11.

**Proposition 4.** If \( X \) is compact, then \( u \) is a homeomorphism of \( X \) onto itself.

**Proof.** Since \( \tau \) is jointly continuous, \( \tau \) is finer than the compact-open topology \( \tau_e \) on \( C(X) \). Hence \( u_\alpha \) converges to \( u \) under \( \tau_e \). \( \{ u_\alpha^{-1} \} \) is relatively compact in \( C(X) \) under \( \tau_e \) by Lemma 1. Choose a \( \tau_e \)-convergent subnet \( \{ u_{\alpha'}^{-1} \} \) of \( \{ u_\alpha^{-1} \} \), and let \( v \in C(X) \) be the limit. Then both \( uv(x) \) and \( vu(x) \) coincide with \( x \) for each \( x \in X \) by the joint-continuity of \( \tau_e \).

**Corollary (D. van Dantzig and B. L. van der Waerden [3]).** Let \( X \) be a locally compact, metric space that contains but a finite number of connected components, or a compact metric space. Then the family of all isometries of \( X \) onto itself is a locally compact transformation group acting on \( X \) under any topology finer than the point-open topology.

**Proof.** This follows from Propositions 1, 2, 4, and the fact that the point-open topology coincides with the compact-open topology on an equicontinuous family of mappings. Compact case is not treated in [3].

**References**


