Some properties of pullback diagrams and pushout diagrams in abelian categories

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Citation

Issue Date

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Some Properties of Pullback Diagrams and Pushout Diagrams in Abelian Categories.

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(Received October 30, 1971)

Abstract.

The following results for pullback diagrams and pushout diagrams in abelian categories were obtained.

Proposition 1. In abelian categories, consider a commutative diagram

\[
\begin{array}{ccc}
P & \rightarrow & B \\
\downarrow & & \downarrow \\
A & \rightarrow & C
\end{array}
\quad (1)
\begin{array}{ccc}
P' & \leftarrow & P' \\
\downarrow & & \downarrow \\
A' & \leftarrow & C
\end{array}
\]

where two squares are pullback diagrams and two morphisms \(A \rightarrow C\) and \(A' \rightarrow C\) are monomorphic. Then each of the following statements is sufficient to ensure that \(P\) and \(P'\) are isomorphic.

(a). \(A \rightarrow C\) and \(A' \rightarrow C\) are equivalent.
(b). \(A' \rightarrow C\) represents the image of \(P \rightarrow A \rightarrow C\).
(c). \(P \rightarrow A\) is epimorphic, and the image of \(P \rightarrow A \rightarrow C\) is equal to the image of \(P' \rightarrow A' \rightarrow C\).
(d). \(B \rightarrow C\) is epimorphic, and the image of \(P \rightarrow A \rightarrow C\) is equal to the image of \(P' \rightarrow A' \rightarrow C\).

Proposition 1*. In abelian categories, consider a commutative diagram.

\[
\begin{array}{ccc}
A & \leftarrow & C \\
\downarrow & & \downarrow \\
P & \leftarrow & B
\end{array}
\quad \begin{array}{ccc}
A' & \rightarrow & C' \\
\downarrow & & \downarrow \\
P' & \rightarrow & B
\end{array}
\]

where two squares are pushout diagrams and two morphisms \(C \rightarrow A\) and \(C \rightarrow A'\) are epimorphic. Then each of the following statements is sufficient to ensure that \(P\) and \(P'\) are isomorphic.

(a*). \(C \rightarrow A\) and \(C \rightarrow A'\) are equivalent.
(b*). \(C \rightarrow A'\) represents the coimage of \(C \rightarrow A \rightarrow P\).
(c*). \(A \rightarrow P\) is monomorphic, and the coimage of \(C \rightarrow A \rightarrow P\) is equal
to the coimage of $C \to A' \to P'$.

(d*). $C \to B$ is monomorphic, and the coimage of $C \to A \to P$ is equal to the coimage of $C \to A' \to P'$.

§1. Introduction

Pullback diagrams and pushout diagrams have fundamental and important roles in theory of categories. For example:

1. In abelian categories, the difference kernel of two morphisms $A \to B$ and $A \to C$ is the unique morphism $P \to A$ such that

$$
\begin{array}{ccc}
P & \to & A \\
\downarrow & & \downarrow \\
(1,x) & \to & (1,y) \\
A & \to & A \times B
\end{array}
$$

is pullback diagram, where $(x, y)$ means the cartesian product of the pair of morphisms $x$ and $y$.

2. On a pullback diagram

$$
\begin{array}{ccc}
P & \to & B \\
\downarrow & & \downarrow \\
A & \to & C
\end{array}
$$

in abelian categories, if $A \to C$ and $B \to C$ are monomorphisms then $P$ is the intersection of the two subobjects $A \to C$ and $B \to C$.

3. If a commutative diagram

$$
\begin{array}{ccc}
A & \to & B \\
\downarrow & \downarrow & \downarrow \\
B & \to & C
\end{array}
$$

is pullback, then $A \to B$ is isomorphic and $B \to C$ is monomorphic. (7.8.10 in [1]).

4. The condition that a category $\mathbb{C}$ has a pullback diagram is equivalent to each of the following conditions.

(a). $\mathbb{C}$ is finite complete.

(b). $\mathbb{C}$ has finite products and difference kernels.

(c). $\mathbb{C}$ has finite products and intersections. 7.8.8 Zatz. in [1].

5. The condition that a category $\mathbb{C}$ has pullback diagrams and pushout diagrams and is normal and conormal is equivalent to each of the following conditions.

(a). $\mathbb{C}$ is an abelian category.

(b). $\mathbb{C}$ has kernels, cokernels, finite products, finite coproducts, and is normal and conormal. (Theorem 20.1. in [2]).

Similar discussion can be applied to pushout diagrams, because pushoutness is dual to pullbackness.

The proposition 2. 151. in [3] says that if two pullback diagrams
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\[
\begin{array}{ccc}
P & \rightarrow & B \\
\downarrow & & \downarrow \\
A & \rightarrow & C \\
\end{array}
\quad
\begin{array}{ccc}
P' & \rightarrow & B' \\
\downarrow & & \downarrow \\
A' & \rightarrow & C' \\
\end{array}
\]

satisfy \( A = A', B = B' \) and \( C = C' \), then \( P \) and \( P' \) are isomorphic. In this paper we study the conditions which satisfy that \( P \) and \( P' \) are isomorphic, under the conditions that \( A = A' \) is omitted.

\section{Definitions and notations.}

\( \rightarrow \) means a \textbf{morphism} from \( A \) to \( B \), where \( A \) and \( B \) are objects, and if there is no confusion we write \( A \rightarrow B \) instead of \( A \rightarrow B \).

\( (A \rightarrow B) \) means the \textbf{equivalence class} represented by \( A \rightarrow B \).

\( \rightarrow \) means the \textbf{zero morphism} from \( A \) to \( B \).

\( 1 \) means the \textbf{identity morphism} from \( A \) to \( A \).

\( \text{Ker} (A \rightarrow B) \) means the \textbf{kernel} of \( A \rightarrow B \): that is the morphism such that

\begin{align*}
K 1. & \quad K \rightarrow A \rightarrow B = K \rightarrow B \\
K 2. & \quad \text{For all } X \rightarrow A \text{ such that } X \rightarrow A \rightarrow B = X \rightarrow B, \text{ there is a unique } X \rightarrow A \text{ such that } X \rightarrow K \rightarrow A = X \rightarrow A.
\end{align*}

\( \text{Cok} (A \rightarrow B) \) means the \textbf{cokernel} of \( A \rightarrow B \): that is the morphism \( B \rightarrow F \) such that

\begin{align*}
C 1. & \quad A \rightarrow B \rightarrow F = A \rightarrow F. \\
C 2. & \quad \text{For all } B \rightarrow X \text{ such that } A \rightarrow B \rightarrow K = A \rightarrow X, \text{ there is a unique } F \rightarrow X \text{ such that } B \rightarrow F \rightarrow X = B \rightarrow X.
\end{align*}

\( \text{Im} (A \rightarrow B) \) means the \textbf{image} of \( A \rightarrow B \): that is the smallest subobject of \( B \) such that \( A \rightarrow B \) factors through the representing monomorphisms.

If a category \( \mathcal{A} \) satisfies the following statements, \( \mathcal{A} \) is called an \textbf{abelian category}.

\begin{enumerate}
\item \( \mathcal{A} \) has a zero object.
\item \( \mathcal{A} \) has a product.
\item \( \mathcal{A} \) has a sum.
\item \( \mathcal{A} \) has a kernel.
\item \( \mathcal{A} \) has a cokernel.
\item \( \mathcal{A} \) has a kernel of a morphism.
\item \( \mathcal{A} \) has a cokernel of a morphism.
\end{enumerate}

A commutative diagram

\[
\begin{array}{ccc}
P & \rightarrow & B \\
\downarrow & & \downarrow \\
A & \rightarrow & C \\
\end{array}
\]

is called a \textbf{pullback diagram} if for every pair of morphisms \( X \rightarrow A \) and \( X \rightarrow B \).
\[ \rightarrow B \text{ such that} \]
\[
\begin{array}{c}
X \rightarrow B \\
\downarrow \downarrow \\
A \rightarrow C
\end{array}
\]

commutes, there is a unique \( X \rightarrow P \) such that
\[
\begin{array}{c}
X \rightarrow P \rightarrow A = X \rightarrow A \\
X \rightarrow P \rightarrow B = X \rightarrow B.
\end{array}
\]

A commutative diagram
\[
\begin{array}{c}
A \rightarrow B \\
\downarrow \downarrow \\
C \rightarrow P
\end{array}
\]
is called pushout diagram if for every pair of morphisms \( X \rightarrow A \) and \( X \rightarrow B \) such that
\[
\begin{array}{c}
A \rightarrow B \\
\downarrow \downarrow \\
C \rightarrow X
\end{array}
\]
commutes, there is a unique \( P \rightarrow X \) such that:
\[
\begin{array}{c}
B \rightarrow P \rightarrow X = B \rightarrow X \\
C \rightarrow P \rightarrow X = C \rightarrow X
\end{array}
\]

A category \( \mathcal{C} \) is called an exact category, if any morphism \( A \rightarrow B \) of \( \mathcal{C} \)
can be written as a composition \( A \rightarrow I \rightarrow B \) where \( q \) is an epimorphism and \( v \) is a monomorphism.

\[ \S 3. \text{ The proof of the propositions.} \]

In the case (a). Since \( A \rightarrow C \) and \( A' \rightarrow C \) are equivalent, there are two morphisms \( A \rightarrow A' \) and \( A' \rightarrow A \) such that
\[
\begin{array}{c}
A \overset{a}{\rightarrow} A' \overset{b}{\rightarrow} C = A \rightarrow C \\
A' \overset{a}{\rightarrow} A \overset{b}{\rightarrow} C = A' \rightarrow C \\
A \overset{a}{\rightarrow} A' \overset{b}{\rightarrow} C = A \rightarrow A', \quad A' \overset{a}{\rightarrow} A \overset{b}{\rightarrow} A' \rightarrow A
\end{array}
\]
hold. Therefore
\[
\begin{array}{c}
P \overset{a}{\rightarrow} A \overset{a}{\rightarrow} A' = C = P \overset{a}{\rightarrow} A \rightarrow C
\end{array}
\]
and from the commutativity of the diagram (1)
\[
\begin{array}{c}
P \overset{a}{\rightarrow} A \overset{a}{\rightarrow} C = P \rightarrow B \rightarrow C
\end{array}
\]
follows. By (2), and (3) we can find a unique \( x \)
\[
\begin{array}{c}
P \overset{x}{\rightarrow} P' \rightarrow A' = P \rightarrow A \rightarrow A' \\
\end{array}
\]
\[
\begin{array}{c}
P \overset{x}{\rightarrow} P' \rightarrow B = P \rightarrow B
\end{array}
\]
because the right-hand square is pullback.

Similarly we obtain a unique \( P' \rightarrow P \) such that
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\[ P' \xrightarrow{y} P \rightarrow A = P' \rightarrow A' \rightarrow A \]  \quad (6)
\[ P' \rightarrow P \rightarrow B = P' \rightarrow B \]  \quad (7)

By (4) and (6) we obtain
\[ P \xrightarrow{x} P' \rightarrow P \rightarrow A = P \xrightarrow{x} P' \rightarrow A' \rightarrow A \]
\[ P \rightarrow A \rightarrow A' \rightarrow A = P \rightarrow A \rightarrow A = P \rightarrow A \]

and by (5) and (7) we obtain
\[ P \xrightarrow{y} P' \rightarrow P \rightarrow B = P \xrightarrow{y} P' \rightarrow B = P \rightarrow B \]

On the other hand we have
\[ P \xrightarrow{1} P \rightarrow A = P \rightarrow A \]
\[ P \xrightarrow{1} P \rightarrow B = P \rightarrow B \]

By the uniqueness of the morphism \( P \rightarrow P' \) such that
\[ P \xrightarrow{1} P \rightarrow A = P \rightarrow A \]
\[ P \xrightarrow{1} P \rightarrow B = P \rightarrow B \]

we have
\[ P \xrightarrow{y} P' \rightarrow P = P \xrightarrow{1} P \]  \quad (8)

Similarly we have
\[ P' \xrightarrow{y} P \rightarrow P' = P' \xrightarrow{1} P' \]  \quad (9)

By (8) and (9) \( P \) and \( P' \) are isomorphic.

In the case (b). Let \( C \rightarrow F \) be a cokernel of \( P \rightarrow B \rightarrow C \) (The existence of a cokernel is ensured by the assumption that the category is abelian).

Then by the theorem 2. 16. in [3] and the assumption that
\[ \langle A' \rightarrow C \rangle = \text{Im}(P \rightarrow B \rightarrow C) \]

we obtain
\[ A' \rightarrow C = \text{Ker}(C \rightarrow F) \]
and therefore we can find a morphism \( P \rightarrow A' \) such that
\[ P \rightarrow A' \rightarrow C = P \rightarrow B \rightarrow C \]  \quad (10)

By the assumption that the right-hand square is pullback, there is a unique morphism \( P \rightarrow P' \) such that
\[ P \rightarrow P' \rightarrow A' = P \rightarrow A' \]  \quad (11)
\[ P \rightarrow P' \rightarrow B = P \rightarrow B . \]  \quad (12)

Let \( C \rightarrow F' \) be a cokernel of \( A \rightarrow C \), then \( P \rightarrow B \rightarrow C \rightarrow F' \) vanishes and by \( C \rightarrow F = \text{Cok}(P \rightarrow B \rightarrow C) \), there is a morphism \( F \rightarrow F' \) such that the diagram
\[ C \rightarrow F \]
\[ \begin{array}{c}
F' \\
\text{commues}
\end{array} \]

Hence \( A' \rightarrow C \rightarrow F' = A' \rightarrow C \rightarrow F \rightarrow F' \) vanishes, and there is a morphism
A' → A such that the diagram
\[
\begin{array}{ccc}
A' & \to & A \\
\downarrow & & \downarrow \\
C & \to & \text{commutes,}
\end{array}
\]

By (10) and the commutativity of (13) we have
\[
P \to A' \to A \to C = P \to A' \to C
\]
\[
= P \to B \to C = P \to A \to C
\]
and the fact that A → C is monomorphic yields the commutativity of the diagram
\[
P \to A'
\]
\[
\begin{array}{ccc}
\downarrow & & \\
A & \to & \text{commutes,}
\end{array}
\]

By (14) and the pullbackness of the left-hand square, we can find a unique morphism \( P' \to P \) such that
\[
P' \to P \to B = P' \to B
\]
\[
P' \to P \to A = P' \to A' \to A
\]
Now by the commutativity of (13) and the equations (10) and (13) we have
\[
P \to P' \to P \to A = P \to P' \to A' \to A
\]
\[
P \to P' \to P \to B = P \to P' \to B = P \to B.
\]
On the other hand
\[
P' \to P \to A = P \to A
\]
\[
P' \to P \to B = P \to B
\]
hold, therefore by the uniqueness of the morphism \( P \to P \) such that
\[
P \to P \to A = P \to A
\]
\[
P \to P \to B = P \to B
\]
we obtain
\[
P \to P' \to P = P \to P
\]
\[
P' \to P \to P' = P' \to P'.
\]
this shows that \( P \) and \( P' \) are isomorphic.

In the case (c), Put \( \text{Cok}(P \to A \to C) = C \to F \) and \( \text{Cok}(A \to C) = C \to F \), then
\[
P \to A \to C \to F = 0
\]
\[
A \to C \to F' = 0.
\]
hold. By (16) and the fact that \( P \to A \) is epimorphic, we obtain
\[
A \to C \to F = 0.
\]
Therefore we can find a morphism \( F \to F' \) such that
\[
\begin{array}{ccc}
C & \to & F \\
\downarrow & & \downarrow \\
F' & \to & \text{commutes,}
\end{array}
\]
and by \( \text{Cok}(P \to A \to C) = C \to F \) and (17) we can find a morphism \( F' \to F \).
such that
\[
\begin{array}{c}
C \rightarrow F' \\
\downarrow \\
F \\
\end{array}
\]
commutes.
These show that \( \text{Cok}(P \rightarrow A \rightarrow C) = \text{Cok}(A \rightarrow C) \) holds.
Therefore we have
\[
\text{Im}(P \rightarrow A \rightarrow C) = [\text{ker}\text{Cok}(P \rightarrow A \rightarrow C)] = [\text{Ker}\text{Cok}(A \rightarrow C)] = [A \rightarrow C]
\]
where the last equality follows from the fact \( A \rightarrow C \) is a kernel of a morphism (since \( A \rightarrow C \) is monomorphic and we are considering in abelian categories), and Hilfssatz 1 in \([4]\). By \((8)\) and the assumption we obtain
\[
\text{Im}(P \rightarrow A \rightarrow C) = \text{Im}(P' \rightarrow A' \rightarrow C)
\]
hence we have
\[
(A \rightarrow C) = \text{Im}(P \rightarrow A \rightarrow C).
\]
These show that the assumption of \((c)\) covers it of \((b)\).

In the case \((d)\). Hilfssatz 3 in \([4]\) says that if a category \( \mathcal{C} \) satisfies \(A1, A2, A2^*, A3, A3^*\), then \( \mathcal{C} \) is an exact category. Therefore our category is exact. Corollary 16.3 in \([2]\) says that in an exact category if a commutative diagram
\[
\begin{array}{c}
B' \rightarrow C' \\
\downarrow \\
B \rightarrow C
\end{array}
\]
is pullback, \( B \rightarrow C \) is epimorphic and \( C' \rightarrow C \) is monomorphic, then this can be extended to a commutative diagram
\[
\begin{array}{c}
O \\
\downarrow \\
O \rightarrow A \rightarrow B' \rightarrow C' \rightarrow O \\
\downarrow \\
O \rightarrow A \rightarrow B \rightarrow C \rightarrow O \\
\downarrow \\
C'' = C'' \\
\downarrow \\
O \rightarrow O
\end{array}
\]
with exact rows and columns. This diagram shows that \( B' \rightarrow C' \) is epimorphic. By the above statement and the assumption of proposition \(1\) we obtain that \( P \rightarrow A \) in \((1)\) is epimorphic. Therefore the assumption of \((d)\) covers it of \((c)\). Thus the proposition \(1\) is proved.

The proof of the proposition \(1^*\). Pushoutness is dual to pullbackness and also the assumption of proposition \(1^*\) is dual to the assumption of proposition \(1\). Therefore if we substitute the terms in the proof of proposition \(1\) for the dual terms of them, then the proof of the proposition \(1^*\) is finished.
Some Properties of Pullback Diagrams and Pushout Diagrams in Abelian Categories.

Reference