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<td>Citation</td>
<td>長崎大学教育学部自然科学研究報告</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1972-02-29</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10069/33016">http://hdl.handle.net/10069/33016</a></td>
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A Note on k-Modules in an Algebraic Function Field \( K/k \) of One Variable

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(Received October 31, 1971)

§ 1. Abstract.

Let \( K \) be an algebraic function field of one variable with a constant field \( k \). It is not necessary, in this note, that \( k \) is the exact constant field of \( K \). We shall denote by \( M \) a finitely generated \( k \)-module in \( K \).

Moreover \( n(M) \) denotes the denominator divisor of \( M \), i.e., \( n(M) \) is the divisor of \( K \) defined by

\[
\text{ord}_p n(M) = \max \{ \text{ord}_p n(x) \}
\]

\[
\text{ord}_x \neq 0
\]

for every prime divisor \( p \) of \( K \), where \( \text{ord}_p \) denotes the order at \( p \) and \( n(x) \) means the denominator divisor of \( x \). Then it is well-known that there exists some element \( x \) in \( M \) such that \( n(M) = n(x) \), if \( k \) contains enough elements. (e.g., E. Artin [1]; p. 318, Lemma 2).

The purpose of this note is to study it in minute detail. Under the condition \( d(n(M)) \leq |k| \), we shall prove that there exists an element \( x \) in \( M \) such that \( n(M) = n(x) \) in § 2, where \( d(n(M)) \) means the degree of \( n(M) \) and \( |k| \) denotes the number of all the elements contained in \( k \); if \( k \) is not finite, then we shall put \( |k| = \infty \). In § 3, we shall show that the above inequality is the best condition in a sense.

§ 2. Sufficient condition.

The notations being as in § 1, we shall prove the following theorem.

**Theorem.** If the inequality \( d(n(M)) \leq |k| \) holds, there exists some element \( x \) in \( M \) satisfying the equality \( n(M) = n(x) \).
Proof. In order to prove the theorem, we shall use the induction with respect to the dimension \( l \) of \( M \). In the case of \( l=1 \), there exists an element \( x \) in \( M \) satisfying \( M=k \cdot x \), where \( k \cdot x \) means the \( k \)-module generated by \( x \). If we take such an element \( x \), we have \( n(M)=n(x) \) obviously.

In the case of \( l \geq 2 \), we assume that the theorem is correct for every \( k \)-module \( M \) whose dimension is at most \( l-1 \). We can take a \( k \)-module \( \widetilde{M} \) of dimension \( l-1 \) in \( M \). So we shall take such a \( k \)-module \( \widetilde{M} \). Then there exists an element \( x_2 \) in \( M \) such that \( M=\widetilde{M}+k \cdot x_2 \). We shall take such an element \( x_2 \).

Since \( n(\widetilde{M}) \mid n(M) \) i.e., \( n(\widetilde{M}) \) divides \( n(M) \), the inequalities \( d(n(\widetilde{M})) \leq d(n(M)) \leq k \) hold trivially. So there exists an element \( x_1 \) in \( M \) such that \( n(\widetilde{M})=n(x_1) \) by the assumption of the induction.

This implies that \( n(\widetilde{M}) \) is the least common multiple of \( n(x_1) \) and \( n(x_2) \). For the equality \( M=\widetilde{M}+k \cdot x_2 \) means that \( n(M) \) is the least common multiple of \( n(\widetilde{M}) \) and \( n(x_2) \). By making use of this result, we shall show the existence of the elements \( a \) and \( b \) in \( k \) such that \( n(M)=n(ax_1+bx_2) \).

If \( n(x_1) \mid n(x_2) \) or \( n(x_2) \mid n(x_1) \), then \( n(M)=n(x_2) \) or \( n(M)=n(x_1) \) holds. If \( n(x_1) \) and \( n(x_2) \) are coprime, then \( n(M)=n(x_1+x_2) \) holds. Therefore it is enough for us to consider \( M \) only in the case that

\[
\begin{aligned}
(1) & \quad n(x_1)+n(x_2), \\
(2) & \quad n(x_2) \mid n(x_1) \quad \text{and} \\
(3) & \quad n(x_1) \text{ and } n(x_2) \text{ are not coprime}.
\end{aligned}
\]

Now we shall take a factorization of \( n(M) \) into prime divisors and we shall denote it by \( n(M)=\prod_{i=1}^{r} p_i^{e_i} \), where \( e_i \) is the integer \( \geq 1 \) for \( i=1, \ldots, r \). Moreover we assume that \( p_1, \ldots, p_m \) consist of all prime divisors \( p_i \) of \( K \) such that \( \text{ord}_{p_i} n(x_1)=\text{ord}_{p_i} n(x_2) \) holds. Obviously \( m \geq 1 \). Since (1),

\[
(2) \quad m+2 \leq r \leq d(n(M)) \text{ holds.}
\]

We shall denote by \( t_i \) a prime element of \( p_i \). Then obviously \( \text{ord}_{p_i} (x, t_i^{e_i}) = \text{ord}_{p_i} (x_2 t_i^{e_i}) = 0 \) for \( i=1, \ldots, m \). Moreover we shall denote by \( (x, t_i^{e_i})_p \) and \( (x_2 t_i^{e_i})_p \) the residue classes mod. \( p_i \) of \( x, t_i^{e_i} \) and \( x_2 t_i^{e_i} \) respectively. Then they are finite and are not 0.

Since the assumption of the theorem and (2), \( k \) has at least \( m+2 \) elements, whence we can take an element \( a \) in \( k \) such that \( (x, t_i^{e_i})_p = a (x_2 t_i^{e_i})_p \) for \( i=1, \ldots, m \) and \( a \neq 0 \). We shall take such an element \( a \) in \( k \). Then \( (x, t_i^{e_i}) \) and \( (x_2 t_i^{e_i})_p \) are finite and is not 0, \( (i=1, \ldots, m) \). This gives

\[
(3) \quad \text{ord}_{p_i} (x_i-a x_2) = -e_i \quad \text{for } i=1, \ldots, m.
\]

On the other hand, from \( a \neq 0 \),

\[
(4) \quad \text{ord}_{p_i} (x_i-a x_2) = \min \{ \text{ord}_{p_i} x_i, \text{ord}_{p_i} x_2 \} = -e_i \quad \text{for } i=m+1, \ldots, r.
\]
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Since every prime divisor other than $p_1, \ldots, p_r$ does not divide $n(x_1-ax_2)$, the equalities (3) and (4) imply $n(x_1-ax_2)=n(M)$. The theorem is thereby proved.

§ 3. Consideration for the condition.

The assumption $d(n(M)) \leq |k|$ in the theorem is the best condition for $M$ in a sense. In fact, it happens that we can not find an element $x$ for a $k$-module $M$ satisfying $d(n(M))=|k|+1$ such that $n(M)=n(x)$. So we shall show such an example.

We shall consider $K=k(x,y)$ satisfying $k=GF(5)$ and $y^2=x(x-1)(x-2)(x-3)(x-4)$. Let $M$ be the $k$-module generated by $x_1=x/y$ and $x_2=(x-1)^2/y$.

Then, in order to investigate the denominator divisor of an element of the form $ax_1+bx_2$, $(a \in k, b \in k)$, it is sufficient for us to do only for six elements $x_1, x_2, x_1+x_2, x_1+2x_2, x_1+3x_2$ and $x_1+4x_2$.

We shall denote by $p_a$ the prime divisor of $K$ which divides the numerator divisor of $x-a$ for $a=0, 1, 2, 3, 4$. $p$ denotes the prime divisor of $K$ which divides $n(x)$. Then we obtain easily $\text{ord } p_1 x_1=\text{ord } p_2 x_2=5$, $\text{ord } p_3 (x_1+x_2)=\text{ord } p_4 (x_1+3x_2)=\text{ord } p_5 (x_1+4x_2)=1$ and $n(M)=p_1p_2p_3p_4p_5$.

Whence we can not find a $k$-linear combination of $x/y$ and $(x-1)^2/y$ whose denominator divisor is $n(M)$. In this case, $|k|=5$ and $d(n(M))=6$ i.e., $d(n(M))=|k|+1$ holds.

Bibliography