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Expression of the epuations of motion of air in spherical polar-and cylindrical coordinate.

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Introduction

The equations of motion of a particle of mass in a system of three Cartesian coordinates are well familiar to us with its origin situating at the earth's center or on its surface.

The equations of motion of air and that of continunity of a fluid in the above defined coordinate system can be most readily derived by considering the pressure of fluid.

The author has contrived a simple and convenient method of expressing the corresponding form of equations referred to the spherical polar-and cylindrical coordinates with the preceding origin.

Now, take axes x, y and z at p, at the earth's surbace the axis of x being horizontal, and directed towards South, the axis of y being horizontal and directed towards East, and the axis of z being vertical. The directions of these axes are changing with the motion of p. Let u, v, w be component velocities along these three axes, x, y, z be the component of external force, p, ρ be respectively the pressure and density of the air and ω and φ be the angular velocity of the earth's rotation and latitude of P. Then the equations of motion of air are as follows.

$$\frac{du}{dt}-2\,\omega\,v\,sin\varphi=X-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (1)$$

$$\frac{dv}{dt}+2\,\omega(u\,sin\varphi+w\,cos\varphi)=Y-\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (2)$$

$$\frac{dw}{dt}-2\,\omega\,v\,cos\varphi=Z-g-\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (3)$$

The relations between the Cartesian and cylindrical polar coordinates are

$$x=r\,cos\theta\,,\quad y=r\,sin\theta\,,\quad z=z,\quad t=t,$$

Let γ, ξ, w be the component velocities of r, θ and z directions then we have

$$\gamma = u\cos\theta + v\sin\theta,\quad u = \gamma\cos\theta - \xi\sin\theta,\quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\sin\theta}{r}\,\frac{\partial u}{\partial \theta}$$

$$\xi = v\cos\theta - u\sin\theta,\quad v = \gamma\sin\theta + \xi\cos\theta,\quad \frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial u}{\partial \theta}$$

$$w = w \tag{5}$$

Hence we have

$$u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \gamma\frac{\partial u}{\partial r} + \frac{\xi}{r}\,\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} \tag{6}$$

In the same way

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \gamma\frac{\partial v}{\partial r} + \frac{\xi}{r}\,\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} \tag{7}$$

and

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial \gamma}{\partial t}\cos\theta - \frac{\partial \xi}{\partial t}\sin\theta,\quad \frac{\partial p}{\partial x} = \cos\theta\frac{\partial p}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial p}{\partial \theta},\\ &\frac{\partial v}{\partial t} = \frac{\partial \gamma}{\partial t}\sin\theta + \frac{\partial \xi}{\partial t}\cos\theta,\quad \frac{\partial p}{\partial y} = \sin\theta\frac{\partial p}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial p}{\partial \theta} \end{aligned}\right\} \tag{8}$$

Let F_r, F_θ, F_z, be the component of external force along r, θ, z, so

$$\left.\begin{aligned} &X = F_r\cos\theta - F_\theta\sin\theta,\\ &Y = F_r\sin\theta + F_\theta\cos\theta,\\ &Z = F_z \end{aligned}\right\} \tag{9}$$

Then equation (1) can be transformed to (10) using relation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\frac{\partial \gamma}{\partial t}\cos\theta - \frac{\partial \xi}{\partial t}\sin\theta + \gamma\left(\frac{\partial \gamma}{\partial r}\cos\theta - \frac{\partial \xi}{\partial r}\sin\theta\right) + \frac{\xi}{r}\left(\frac{\partial \gamma}{\partial \theta}\cos\theta - \gamma\sin\theta - \right.$$

$$\left.\frac{\partial \xi}{\partial \theta}\sin\theta - \xi\cos\theta\right) + w\left(\frac{\partial \gamma}{\partial z}\cos\theta - \frac{\partial \xi}{\partial z}\sin\theta\right) - 2\,\omega\sin\varphi(\gamma\sin\theta +$$

$$\xi\cos\theta) = F_r\cos\theta - F_\theta\sin\theta - \frac{1}{\rho}\left(\frac{\partial p}{\partial r}\cos\theta - \frac{\partial p}{\partial \theta}\,\frac{\sin\theta}{r}\right) \tag{10}$$

In the same way equation (2) will become

$$\frac{\partial \gamma}{\partial t}\sin\theta + \frac{\partial \xi}{\partial t}\cos\theta + \gamma\left(\frac{\partial \gamma}{\partial r}\sin\theta + \frac{\partial \xi}{\partial r}\cos\theta\right) + \frac{\xi}{r}\left(\frac{\partial \gamma}{\partial \theta}\theta\sin + \gamma\cos\theta + \right.$$

$$\left.\frac{\partial \xi}{\partial \theta}\cos\theta - \xi\sin\theta\right) + w\left(\frac{\partial \gamma}{\partial z}\theta\sin + \frac{\partial \xi}{\partial z}\cos\theta\right) + 2\,\omega(\gamma\cos\theta\sin\varphi - \xi\sin\theta$$

$$\sin\varphi + w\cos\varphi) = F_r\sin\theta + F_\theta\cos\theta - \frac{1}{\rho}\left(\frac{\partial p}{\partial r}\sin\theta + \frac{\cos\theta}{\gamma}\,\frac{\partial p}{\partial \theta}\right. \tag{11}$$

By (10) $\times\cos\theta$ + (11) $\times\sin\theta$, we have

$$\frac{\partial \gamma}{\partial t} + \gamma\frac{\partial \gamma}{\partial r} + \frac{\xi}{r}\left(\frac{\partial \gamma}{\partial \theta} - \xi\right) + w\frac{\partial \gamma}{\partial z} - 2\,\omega\,\xi\sin\varphi + 2\,\omega\,w\cos\varphi\sin\theta = F_r -$$

$$\frac{1}{\rho}\,\frac{\partial p}{\partial r} \tag{12}$$

And by (11) $\times\sin\theta$ $-$(10) $\times\cos\theta$, we have

$$\frac{\partial \xi}{\partial t}+r\frac{\partial \xi}{\partial r}+\frac{\xi}{r}\left(\frac{\partial \xi}{\partial \theta}+r\right)+w\frac{\partial \xi}{\partial z}+2\,\omega(r\sin\varphi+w\cos\varphi\cos\theta)=F_{\theta}-$$

$$\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} \qquad (13)$$

Equation (3) can be directly transformed to

$$\frac{\partial w}{\partial t}+r\frac{\partial w}{\partial r}+\frac{\xi}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial w}{\partial z}-2\,w\cos\varphi\,(r\sin\theta+\xi\cos\theta)=F_{z}-g-$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (14)$$

Next we will proceed to the problem of transformation from the Cartesian to the spherical polar coordinates neglecting the deviating force due to the earth's rotation.

we have the relation in this case

$$x=r\sin\theta\cos\varphi,\quad y=r\sin\theta\sin\varphi,\quad z=r\cos\theta \qquad (15)$$

Let ξ, η, ζ be the component velocities along the directions of r, θ, φ. Let us take three axes x', y' and z' along the above three directions, then we have the next relation between both systems (x, y, z) and (x', y', z')

	x'	y'	z'
x	$\sin\theta\cos\varphi$	$\cos\theta\cos\varphi$	$-\sin\varphi$
y	$\sin\theta\sin\varphi$	$\cos\theta\sin\varphi$	$\cos\varphi$
z	$\cos\varphi$	$-\sin\theta$	0

(16)

Hence we get

$$\left.\begin{aligned}\xi&=-u\sin\theta\cos\varphi+v\sin\theta\sin\varphi+w\cos\theta\\ \eta&=u\cos\theta\cos\varphi+v\cos\theta\sin\varphi-w\sin\theta\\ \zeta&=u\sin\varphi+\cos\varphi\end{aligned}\right\} \qquad (17)$$

Moreover

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\frac{\partial(u,v,w)}{\partial r}\frac{\partial r}{\partial(x,y,z)}+\frac{\partial(u,v,w)}{\partial \theta}\frac{\partial \theta}{\partial(x,y,z)}+$$

$$\frac{\partial(u,v,w)}{\partial \varphi}\frac{\partial \varphi}{\partial(x,y,z)} \qquad (18)$$

In (18)

$$\left.\begin{aligned}&\frac{\partial r}{\partial x}=\sin\theta\cos\varphi,\quad \frac{\partial \theta}{\partial x}=\frac{\cos\theta\cos\varphi}{r},\quad \frac{\partial \varphi}{\partial x}=-\frac{\sin\varphi}{r\sin\theta}\\ &\frac{\partial r}{\partial y}=\sin\theta\sin\varphi,\quad \frac{\partial \theta}{\partial y}=\frac{\cos\theta\sin\varphi}{r},\quad \frac{\partial \varphi}{\partial y}=\frac{\cos\varphi}{r\sin\theta}\\ &\frac{\partial r}{\partial z}=\cos\theta,\quad \frac{\partial \theta}{\partial z}=-\frac{\sin\theta}{r},\quad \frac{\partial \varphi}{\partial z}=0\end{aligned}\right\} \qquad (19)$$

Eventually we have

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\left(u\frac{\partial r}{\partial x}+v\frac{\partial r}{\partial y}+w\frac{\partial r}{\partial z}\right)\frac{\partial u}{\partial r}+\left(u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}+w\right.$$

$$\frac{\partial\theta}{\partial z}\Big)\frac{\partial u}{\partial\theta}+\Big(u\frac{\partial\varphi}{\partial x}+v\frac{\partial\varphi}{\partial y}+w\frac{\partial\varphi}{\partial z}\Big)\frac{\partial u}{\partial\varphi}=\xi\frac{\partial u}{\partial r}+\frac{\eta}{r}\frac{\partial u}{\partial\theta}+\frac{\zeta}{r\sin\theta}\frac{\partial u}{\partial\varphi} \qquad (20)$$

And

$$\frac{\partial u}{\partial t}=\frac{\partial\xi}{\partial t}\sin\theta\cos\varphi+\frac{\partial\eta}{\partial t}\cos\theta\cos\varphi-\frac{\partial\zeta}{\partial t}\sin\varphi \qquad (21)$$

In this way Euler's equation can be transformed as follows.

$$\frac{\partial\xi}{\partial t}(x\,x')+\frac{\partial\eta}{\partial t}(x\,y')+\frac{\partial\zeta}{\partial t}(x\,z')+\xi\Big(\frac{\partial\xi}{\partial r}(x\,x')+\frac{\partial\eta}{\partial r}(x\,y')+\frac{\partial\zeta}{\partial r}$$

$$(x\,z')\Big)+\frac{\eta}{r}\Big(\frac{\partial\xi}{\partial\theta}(x\,x')+\frac{\partial\eta}{\partial\theta}(x\,y')+\frac{\partial\zeta}{\partial\theta}(x\,z')+\xi(x\,y')-\eta(x\,x')\Big)+$$

$$\frac{\zeta}{r\sin\theta}\Big(\frac{\partial\xi}{\partial\varphi}(x\,x')+\frac{\partial\eta}{\partial\varphi}(x\,y')+\frac{\partial\zeta}{\partial\varphi}(x\,z')-\xi(y\,x')-\eta(y\,y')-\zeta$$

$$(y\,z')\Big)=F_r(x\,x')+F_\theta(x\,y')+F_\varphi(x\,z')-\frac{1}{\rho}\Big(\frac{\partial p}{\partial r}(x\,x')+\frac{1}{r}\frac{\partial p}{\partial\theta}(x\,y')$$

$$+\frac{1}{r\sin\theta}\frac{\partial p}{\partial\varphi}(x\,z')\Big) \qquad (22)$$

$$\frac{\partial\xi}{\partial t}(y\,x')+\frac{\partial\eta}{\partial t}(y\,y')+\frac{\partial\zeta}{\partial t}(y\,z')+\xi\Big(\frac{\partial\xi}{\partial r}(y\,x')+\frac{\partial\eta}{\partial r}(y\,y')+\frac{\partial\zeta}{\partial r}$$

$$(y\,z')\Big)+\frac{\eta}{r}\Big(\frac{\partial\xi}{\partial\theta}(y\,x')+\frac{\partial\eta}{\partial\theta}(y\,y')+\frac{\partial\zeta}{\partial\theta}(y\,z')+\xi(y\,y')-\eta(y\,x')\Big)+$$

$$\frac{\zeta}{r\sin\theta}\Big(\frac{\partial\xi}{\partial\varphi}(y\,x')+\frac{\partial\eta}{\partial\varphi}(y\,y')+\frac{\partial\zeta}{\partial\varphi}(y\,z')+\xi(x\,x')+\eta(x\,y')+\zeta$$

$$(x\,z')=F_r(y\,x')+F_\theta(y\,y')+F_\varphi(y\,z')-\frac{1}{\rho}\Big(\frac{\partial p}{\partial r}(y\,x')+\frac{1}{r}\frac{\partial p}{\partial\theta}(y\,y')+$$

$$\frac{1}{r\sin\theta}\frac{\partial p}{\partial\varphi}(y\,z') \qquad (23)$$

$$\frac{\partial\xi}{\partial t}(z\,x')+\frac{\partial\eta}{\partial t}(z\,y')+\xi\Big(\frac{\partial\xi}{\partial r}(z\,x')+\frac{\partial\eta}{\partial r}(z\,y')\Big)+\frac{\eta}{r}\Big(\frac{\partial\xi}{\partial\theta}(z\,x')+$$

$$\frac{\partial\eta}{\partial\theta}(z\,y')+\xi(z\,y')-\eta(z\,x')\Big)+\frac{\zeta}{r\sin\theta}\Big(\frac{\partial\xi}{\partial\varphi}(z\,x')+\frac{\partial\eta}{\partial\varphi}(z\,y')\Big)=$$

$$F_r(z\,x')+F_\theta(z\,y')-\frac{1}{\rho}\Big(\frac{\partial p}{\partial r}(z\,x')+\frac{1}{r}\frac{\partial p}{\partial\theta}(z\,y')\Big) \qquad (24)$$

By the procedure $(22)\times(x\,x')+(23)\times(y\,x')+(24)\times(z\,x')$
we get

$$\frac{\partial\xi}{\partial t}+\xi\frac{\partial\xi}{\partial r}+\frac{\eta}{r}\Big(\frac{\partial\xi}{\partial\theta}-\eta\Big)+\frac{\zeta}{r\sin\theta}\Big(\frac{\partial\xi}{\partial\varphi}-\zeta\sin\theta\Big)=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (25)$$

Similarly, we get by $(22)\times(x\,y')+(23)\times(y\,y')+(24)\times(z\,y')$

$$\frac{\partial\eta}{\partial t}+\xi\frac{\partial\eta}{\partial r}+\frac{\eta}{r}\Big(\frac{\partial\eta}{\partial\theta}+\xi\Big)+\frac{\zeta}{r\sin\theta}\Big(\frac{\partial\eta}{\partial\varphi}-\zeta\cos\theta\Big)=F_\theta-\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \qquad (26)$$

Moreover we get by (22) × ($x z'$) + (23) × ($y z'$) + (24) × ($z z'$)

$$\frac{\partial \varsigma}{\partial t}+\xi\frac{\partial \varsigma}{\partial r}+\frac{\eta}{r}\frac{\partial \varsigma}{\partial \theta}+\frac{\varsigma}{r\sin\theta}\left(\frac{\partial \varsigma}{\partial \varphi}+\xi\sin\theta+\eta\cos\theta\right)=F_\varphi-\frac{1}{\rho\, r\sin\theta}\frac{\partial \rho}{\partial \varphi} \tag{27}$$

These three formulas reduce to (28), (29), (30) taking account of the total differentiation following the fluid.

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\xi\frac{\partial}{\partial r}+\frac{\eta}{r}\frac{\partial}{\partial \theta}+\frac{\varsigma}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

$$\frac{d\xi}{dt}-\frac{1}{r}(\eta^2+\zeta^2)=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{28}$$

$$\frac{d\eta}{dt}+\frac{\xi\eta}{r}-\frac{\zeta^2\cos\theta}{r}=F_\theta-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \tag{29}$$

$$\frac{d\zeta}{dt}+\frac{\xi\varsigma}{r}+\frac{\eta\varsigma\cos\theta}{r}=F_\varphi-\frac{1}{\rho\, r\sin\theta}\frac{\partial p}{\partial \varphi} \tag{30}$$

Equations (28), (29), (30) are concerning to polar coordinates fixed in space.

When we adopt the Cartesian coordinates (x, y, z), the origin being the earth's center, Z axis being the earth's axis of rotation, and $x\ y$ plane being the earth's equatorial plane, the equations of motion will become

$$\left.\begin{aligned}&\frac{du}{dt}-2\omega v-x\omega^2=X-\frac{1}{\rho}\frac{\partial p}{\partial x}\\&\frac{dv}{dt}+2\omega u-y\omega^2=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}\\&\frac{dw}{dt}=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}\end{aligned}\right\} \tag{31}$$

By means of the relation between the Cartesian and polar coordinates, we get the following equation referred to the latter coordinates.

$$\left.\begin{aligned}&\frac{d\xi}{dt}-\frac{1}{r}(\eta^2+\varsigma^2)-2\omega\varsigma\sin\theta-r\omega^2\sin^2\theta=F_r-\frac{1}{\rho}\frac{\partial p}{\partial r}\\&\frac{d\eta}{dt}+\frac{\xi\eta}{r}-\frac{\varsigma^2\cos\theta}{r}-2\omega\varsigma\cos\theta-r\omega^2\sin\theta\cos\theta=F_\theta-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}\\&\frac{d\varsigma}{dt}+\frac{\xi\varsigma}{r}+\frac{\eta\varsigma\cos\theta}{r}+2\omega\xi\sin\theta+2\omega\eta\cos\theta=F_\varphi-\frac{1}{\rho\, r\sin\theta}\frac{\partial p}{\partial \varphi}\end{aligned}\right\} \tag{32}$$

As the second and third terms on the left-hand side of these equations involve the multiplication of component velocities and therefore produce neglisible small amount.

Let us put $\theta=\frac{\pi}{2}-L$, and replace L for θ, and consequently η for $-\eta$.

Then we have

$$\left.\begin{aligned}
&\frac{d\xi}{dt}-2\,\omega\,\varsigma\,cosL=F_r-g-\frac{1}{\rho}\,\frac{\partial\,p}{\partial\,r}\\
&\frac{d\eta}{dt}+2\,\omega\,\varsigma\,sinL=F_L-\frac{1}{\rho\,r}\,\frac{\partial\,L}{\partial\,p}\\
&\frac{d\varsigma}{dt}+2\,\omega\,(\,\xi\,cosL-\eta\,sinL)=F_\varphi-\frac{1}{\rho\,r\,cosL}\,\frac{\partial\,p}{\partial\,\varphi}
\end{aligned}\right\}\quad(33)$$

here F_r, F_L, F_φ, being the component of external forces other than the gravitation of the earth along the direction of axes.

At last we will proceed to the discussion of equation of continuity.

This equation is expressed by (34) concerning to the Cartesion coordinates

$$\frac{\partial\,\rho}{\partial\,t}+\frac{\partial(\,\rho\,u\,)}{\partial\,x}+\frac{\partial(\,\rho\,v\,)}{\rho\,y}+\frac{\partial(\,\rho\,w)}{\partial\,z}=0 \qquad (34)$$

In cylindrical coordinates it becomes as follows by pursueing the same method and notations.

$$\frac{\partial\,\rho}{\partial\,t}+\frac{1}{r}\,\frac{\partial(\,\rho\,r\,\tau)}{\partial\,r}+\frac{1}{r}\,\frac{\partial(\,\rho\,\xi\,)}{\partial\,\theta}+\frac{\partial(\,\rho\,w)}{\partial\,z}=0 \qquad (35)$$

Similarly it becomes in spherical polar coordinates.

$$\frac{\partial\,\rho}{\partial\,t}+\frac{\partial(\,\rho\,\xi\,)}{\partial\,r}+\frac{1}{r}\,\frac{\partial(\,\rho\,\eta\,)}{\partial\,\theta}+\frac{1}{r\,sin\,\theta}\,\frac{\partial(\,\rho\,\varsigma\,)}{\partial\,\varphi}+\frac{2\,\rho\,\xi}{r}+\frac{cos\,\theta}{r}\,\rho\,\eta=0 \qquad (36)$$

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