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The Phase function in the Tertiary Scattering

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(Manuscript received Oct. 1st in 1967)

Abstract

Let us take at random three points F, T and E in the earth's atmosphere, and a point O' on its surface. Define a system of rectangular coordinates X₁, Y₁, Z₁ with its center at F, with X₁ axis drawn towards the Sun. Let us resolve the direct insolation reaching F into two plane polarized rays: the one travels to X₁ and oscillates in Z₁ direction, which is named by (1), the other travels to X₁ and oscillates in Y₁, which is named by (2). The primarily scattered ray generated at T when (1) encounters one air particle at T travels to FT direction and oscillates in a direction normal to it. FT direction is named by X₂. This direction of oscillation is named by Z₂ and determined by X₁, Y₁, Z₁ and the positions of F and T. This scattering is named by E₁.

The primarily scattered ray generated at T when (2) encounters one air particle at T travels to FT direction and oscillates in a direction normal to FT. This direction is named by Z₂' and determined by X₁, Y₁, Z₁ and F, T. This scattering is named by E₁'.

E₁ travels to TE direction and oscillates in a direction normal to it at the point T. This direction of oscillation is named by Z₃. E₁ scattering generates a secondarily scattered ray at E when it encounters one air particle at E. This secondary scattering is named by E₂. In the same way, E₁' travels to TE direction and oscillates in a direction normal to it at the point T. This direction of oscillation is named by Z₃'. E₁', scattering generates a secondarily scattered ray at E when it encounters one air particle at E. This secondary scattering is named by E₂'.

Further, E₂ and E₂' will generate tertiary scattering at O' when they reach at O' and encounter one air particle at O'.

Let \( \omega, \omega', \omega_2, \omega_2', \omega_3, \omega_3' \) be respectively the angle between FT and Z₁, FT and Y₁, TE and Z₂, TE and Z₂', EO' and Z₃, EO' and Z₃', then the phase function in the tertiary scattering is expressed by

\[
\frac{3}{\pi} \sin^2 \omega_n + \frac{3}{\pi} \sin^2 \omega'_n
\]
Let $O$ be the earth's center. Define a system of rectangular coordinate $X'Y'Z'$ with its center at $O'$. $Z$ is drawn towards $OO'$, $X'$ is in the vertical plane containing the Sun's centre and in the Sun's side. Let $E$, $T$ and $F$ be in the plane $X'Z'$ in the Sun's side and use the notation $\angle EOX' = \theta_1$, $\angle EOT = \theta_2$, $\angle OTF = \theta_3$, $h$ = the Sun's altitude. Then the phase function are calculated as follows in the case when $E$ approaches infinitely to $O'$ with a definite value of $\theta$. $D + D' = 1 + \varphi \sin^2(\theta + \theta_1)$, here $\varphi$ being a function of $	heta_1, \theta_2, \theta_3, h$.

1. Introduction The scattering phenomena of the Sun's ray by the earth's atmosphere is the original cause of the blue sky in the day time and the yellowish and reddish sky in the morning and evening. In the scattering problem the terms scattering and absorption are in general used separately as if they were different from each other, but the latter is originated from the former in the Rayleigh atmosphere. Hence we can reasonably say that the color of the sky is caused only by scattering with no suspicion. The scattering phenomena has been solved at first by Rayleigh by the consideration of dimension (Ref. 1). Then, Max Planck has established the theory of electromagnetic wave and made the theoretical formula of the primary scattering intensity basing on the field equation (Ref. 2).

The author has expanded his research and established the theoretical formula giving the phase function in the secondary scattering and the method of computation and moreover executed the computation precisely (Ref. 3).

2. Theoretical formula Take three points $F$, $T$ and $E$ in the atmosphere and a point $O'$ on the earth's surface (see fig. 1). Define a system of rectangular coordinates $X, Y, Z$, with its centre at $F$, $X_1$ axis drawn towards the Sun. $Z_1$ axis is in the vertical plane passing through the Sun's centre at $O'$. Let $(r, \delta, \kappa)$ be the coordinates of $T$ of this system. Now, if $i$, be the direct insolation reaching $F$, we can resolve it into two plane polarized rays, the one is

$$E(Z_1) = \sqrt{-\frac{i}{2}} \exp(i \theta + imx)$$

which travels to $X_1$, and oscillates in $Z_1$ direction, the other is

$$E(Y_1) = \sqrt{-\frac{i}{2}} \exp(i \theta + imx)$$

travelling to $X_1$ and oscillating in $Y_1$ direction.

Here $\varepsilon = R$, and $\varepsilon$ be the dielectric constant of air particles. Then the equation of wave generated at $T$ when the former ie $E(Z_1)$ encounters one air particle at $T$ is expressed by
The Phase function in the Tertiary Scattering (1)

Fig. 1 Explanation of the polarization angle in the tertiary scattering.

\[
E_{\phi} = -\sqrt{\frac{\pi}{2}} \frac{T}{\lambda} \frac{\bar{e}}{R_1} \frac{\bar{\xi}}{R_1} \exp(i \cdot \text{Im} R_i) \\
E_{\omega} = -\sqrt{\frac{\pi}{2}} \frac{T}{\lambda} \frac{\bar{e}}{R_1} \frac{\bar{\xi}}{R_1} \exp(i \cdot \text{Im} R_i) \\
E_{\tau} = -\sqrt{\frac{\pi}{2}} \frac{T}{\lambda} \frac{\bar{e}}{R_1} \frac{\bar{\xi}}{R_1} \exp(i \cdot \text{Im} R_i)
\]

in which \(\lambda\) and \(T\) are the wavelength and the volume of particle. This wave will proceed into the direction FT and have a direction of oscillation normal to it, and the direction cosine of the direction of oscillation are as follows:

\[
\frac{1}{\sin \omega_1} \frac{\tau_i}{R_i}, \frac{1}{\sin \omega_1} \frac{\kappa_i}{R_i}, \frac{1}{\sin \omega_1} \frac{\delta_i}{R_i}, \frac{1}{\sin \omega_1} \frac{\bar{\xi}_i}{R_i}
\]

(4)
in which \( \sin \omega = \frac{\sqrt{1 + \frac{\delta}{\omega}}}{R_1} \),

i.e. \( \omega \) is the angle between FT and \( Z_1 \) axis. The intensity can be obtained by squaring the amplitude, which is

\[
E_1 = \sqrt{\frac{1}{2}} \frac{\pi T}{R_1} \frac{\alpha}{\lambda^2} e^{\left(\frac{\Delta \alpha}{\epsilon}\right)\sin \omega} \tag{5}
\]

In the same way, the plane polarized light generated at T by \( E \langle Y_1 \rangle \) is

\[
E'_{1z} = \sqrt{\frac{1}{2}} \frac{\pi T}{R_1} \frac{\alpha}{\lambda^2} e^{\left(\frac{\Delta \alpha}{\epsilon}\right)\sin \omega} e^{int-imR_1} \tag{6}
\]

\[
E'_{1y} = -\sqrt{\frac{1}{2}} \frac{\pi T}{R_1} \frac{\alpha}{\lambda^2} e^{\left(\frac{\Delta \alpha}{\epsilon}\right)\sin \omega} e^{int+imR_1} \tag{6}
\]

\[
E'_{1z} = \sqrt{\frac{1}{2}} \frac{\pi T}{R_1} \frac{\alpha}{\lambda^2} e^{\left(\frac{\Delta \alpha}{\epsilon}\right)\sin \omega} e^{int-imR_1} \tag{8}
\]

This will also proceed into FT direction and oscillate normal to it, the amplitude being

\[
E' = \sqrt{\frac{1}{2}} \frac{\pi T}{R_1} \frac{\alpha}{\lambda^2} e^{\left(\frac{\Delta \alpha}{\epsilon}\right)\sin \omega'} \tag{7}
\]

in which \( \omega' \) is the angle between FT and \( Y_1 \) axis. In the following discussion, let the amplitude \( E \) and \( E' \) also be the notation of their corresponding polarizations.

Now take the origin at T, \( X_2 \) axis in TF direction, \( Z_2 \) axis in the direction of oscillation of \( E_1 \), \( Y_2 \) axis normal to \( X_2Z_2 \). In the same way, define \( X_2'Y_2'Z_2' \) with respect to \( E' \), in which \( X_2' \) being identical with \( X_2 \).

Now, take a point \( E \), and let \( TE = \frac{R_2}{R_3} \) and the angular distances of \( TE \) from \( Z_2 \) and \( Z_2' \) axes be \( \omega_2 \) and \( \omega_2' \). Then, \( E_1 \) at \( T \) will also generate one plane polarized light at \( E \). Letting the coordinates of \( E \) referred to \( X_2Y_2Z_2 \) system be \( (\tau_2, \delta_2, \kappa_2) \), then this new plane polarized light will be

\[
E_{2z} = E_1 \frac{\pi T}{R_2} \frac{\alpha}{\lambda^2} e^{\left(\frac{\Delta \alpha}{\epsilon}\right)\frac{\tau_2 \delta_2 \kappa_2}{R_3^2}} e^{int-imR_2} \tag{8}
\]

\[
E_{2v} = E_1 \frac{\pi T}{R_2} \frac{\alpha}{\lambda^2} e^{\left(\frac{\Delta \alpha}{\epsilon}\right)\frac{\tau_2 \delta_2 \kappa_2}{R_3^2}} e^{int+imR_2} \tag{8}
\]

\[
E_{2z} = E_1 \frac{\pi T}{R_2} \frac{\alpha}{\lambda^2} e^{\left(\frac{\Delta \alpha}{\epsilon}\right)\frac{\tau_2 \delta_2 \kappa_2}{R_3^2}} e^{int-imR_2} \tag{8}
\]

whose amplitude being

\[
E_2 = E_1 \frac{\pi T}{R_2} \frac{\alpha}{\lambda^2} e^{\left(\frac{\Delta \alpha}{\epsilon}\right)\sin \omega_2} \tag{9}
\]

In the same way, the plane polarized light generated by \( E' \), at \( E \) point can be
expressed by substituting \( E_2 E'_2 \) for \( E_2 E_1 \) in the above expressions with respect to \( X_a Y_a Z'_a \) system. Here, both directions of oscillation of \( E_2 \) and \( E_2' \) are naturally normal to \( TE \). Now, taking the origin at \( E \), \( X_a \) axis in \( ET \) direction, \( Z_a' \) in the direction of oscillation of \( E_a \), \( Y_a \) normal to them, \( X_a' \) in \( ET \) direction, \( Z_a' \) in the direction of oscillation of \( E_a' \), \( Y_a' \) normal to them, and letting the coordinates of \( O' \) with respect to these systems be \( r'_3 \delta_3 \kappa_3', \delta_3' \kappa_3' \), \( EO'=R_3 \) and the angular distances of \( EO' \) from \( Z_a \) and \( Z_a' \) be \( \omega_3 \) and \( \omega_3' \), then the plane polarized light generated at \( O' \) by \( E_a \) is expressed by referring \( X_a Y_a Z_a \)

\[
E_{a'} = E_a \frac{\pi}{R_3} \frac{T}{\lambda^2} \varepsilon (\Delta \frac{1}{\varepsilon}) \frac{1}{R_i'} \exp(\text{int} - \text{im} R_3)
\]

whose amplitude being

\[
E_a = E_a \frac{\pi}{R_3} \frac{T}{\lambda^2} \varepsilon (\Delta \frac{1}{\varepsilon}) \sin \omega_3
\]

In the same way, the plane polarized light generated by \( E'_2 \) at \( O' \) point can be expressed by substituting \( E'_2 \) for \( E_2 \) in the above expressions with respect to \( X'_a Y'_a Z'_a \) system.

Eventually we can get

\[
E_a = \sqrt{\frac{i}{2}} (\pi \frac{T}{\lambda^2} \varepsilon (\Delta \frac{1}{\varepsilon}))^3 \sin \omega_1 \sin \omega_2 \sin \omega_3
\]

\[
E'_a = \sqrt{\frac{i}{2}} (\pi \frac{T}{\lambda^2} \varepsilon (\Delta \frac{1}{\varepsilon}))^3 \sin \omega'_1 \sin \omega'_2 \sin \omega'_3
\]

Now, let the number of particles in unit volume at \( F \), \( T \) and \( E \) be \( I_F, I_T \) and \( I_E \) respectively and put

\[
k_{mn} = \frac{8 \pi^3 T^3}{3 \lambda^4} I_n \varepsilon^3 (\Delta \frac{1}{\varepsilon})^3
\]

\((n = F, T, E)\)

As the energy is the square of the amplitude, so the intensity of tertiary scattering generated at \( O' \) by unit volume of \( F \), \( T \) and \( E \) will be

\[
\frac{i}{2} \left( \frac{\pi}{\lambda^2} \varepsilon (\Delta \frac{1}{\varepsilon}) \right)^3 I_F' \frac{R'_F}{R'_i} \frac{R'_F}{R'_i} (\sin^2 \omega_1 \sin^2 \omega_2 \sin^2 \omega_3 \sin^2 \omega_1' \sin^2 \omega_2' \sin^2 \omega_3')
\]

\[
= \frac{i}{2} \left( \frac{3}{8 \pi} \right)^3 \frac{1}{R'_i} \frac{1}{R'_i} \left( \frac{\pi}{\lambda^2} \varepsilon \sin^2 \omega_1 + \frac{3}{n=1} \sin^2 \omega_1' \sin^2 \omega_2' \sin^2 \omega_3' \right) k_{x_F} k_{x_T} k_{x_E}
\]

If we consider the effect of absorption on the optical way, we must only
multiply the absorption terms.

3. Practical method of calculation. Let O be the earth's centre and O' a point on its surface. Take a coordinate system X Y Z with its origin at O, Z axis being directed towards OO', X axis normal to Z axis and towards the Sun's side on the plane containing OO' and its centre, Y axis normal to X and Z.

Take a point E in the atmosphere seen from O' and let the coordinates referred to X Y Z system be

\[ X = \text{esin} \tau \cos A, \quad Y = \text{esin} \tau \sin A, \quad Z = \text{ecos} \tau \]

in which

\[ 0 \leq \tau \leq \frac{\pi}{2}, \quad 0 \leq A \leq 2 \pi \]

Let X' Y' Z' system be the parallel translation of X Y Z system by the transformation of the origin O to O', and put (see fig. 2)

\[ X' = \text{Rcos} \theta \cos A, \quad Y' = \text{Rcos} \theta \sin A, \quad Z' = \text{Rsin} \theta \]

![Fig. 2 X'Y'Z' coordinate system and angles A, \theta.](image)

Letting the coordinates of E referred to this new system be X' Y' Z', so we get

\[ X = X', \quad Y = Y', \quad Z = Z' + a^\theta \] (17)

hare a^\theta being the earth's radius.

The relation of (x z z z) coordinate system to (X Y Z) system is defined as follows

<table>
<thead>
<tr>
<th>( x_z )</th>
<th>( y_z )</th>
<th>( z_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>( \cos \tau \cos A )</td>
<td>( -\sin A )</td>
</tr>
<tr>
<td>Y</td>
<td>( \cos \tau \sin A )</td>
<td>( \cos A )</td>
</tr>
<tr>
<td>Z</td>
<td>(-\sin \tau )</td>
<td>0</td>
</tr>
</tbody>
</table>

(18)

Now let the coordinates of T refered to (x z z z) system be

\[ x_z = \text{OTsin} \theta \cos A, \quad y_z = \text{OTsin} \theta \sin A, \quad Z_z \text{rn} = \text{OTcos} \theta \]

(19)

(see fig. 3)
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Let \((x_2', y_2', z_2')\) system be the parallel transformation of \((x_2, y_2, z_2)\) system by the translation of the origin \(O\) to \(E\), and put \(\angle EOT = \theta_2\), \(\angle OET = \theta_1\), then

\[
\sin \theta_2 = \frac{ET}{OT} \sin \theta_1, \tag{20}
\]

and the coordinates of \(T\) referred to \((x_2', y_2', z_2')\) will be

\[
x_2' = ET \sin \theta_1 \cos \alpha_1, \quad y_2' = ET \sin \theta_1 \sin \alpha_1, \quad z_2' = -ET \cos \theta_1, \tag{21}
\]

Giving \(ET, \theta_1, \alpha_1\), we can get \(OT\), so \(\theta_2\) can be found, and eventually \(x_2T, y_2T, z_2T\) i.e. the coordinates of \(T\) referred to \((x_2, y_2, z_2)\) system.

Moreover, let the polar coordinates of \(T\) referred to \((X, Y, Z)\) system be

\[
X_T = OT \sin \rho' \cos \alpha', \quad Y_T = OT \sin \rho' \sin \alpha', \quad Z_T = OT \cos \rho' \tag{22}
\]

(see fig. 4)

The combination of the relation between \((X, Y, Z)\) and \((x_2, y_2, z_2)\) systems (18) and the polar coordinates of \((x_2, y_2, z_2)\) system will generate the following formula.
\[
\begin{align*}
\sin \gamma' \cos \alpha' &= \cos \gamma \cos \alpha \sin \theta + \sin \gamma \sin \alpha \cos \theta \\
\sin \gamma' \sin \alpha' &= \cos \gamma \sin \alpha \sin \theta + \sin \gamma \cos \alpha \cos \theta \\
\cos \gamma' &= -\sin \gamma \sin \alpha \cos \theta + \cos \gamma \cos \alpha \sin \theta
\end{align*}
\] (25)

From this we can find \( \gamma' \) and \( \alpha' \) if the right sides are known.

Define a new system of coordinates \( x_3, y_3, z_3 \) by the next relation, here \( z_3 \) axis is directed to \( OT \) (see fig. 4).

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y )</th>
<th>( Z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos \gamma' \cos \alpha' )</td>
<td>( -\sin \alpha' )</td>
<td>( \sin \gamma' \cos \alpha' )</td>
</tr>
<tr>
<td>( \cos \gamma' \sin \alpha' )</td>
<td>( \cos \alpha' )</td>
<td>( \sin \gamma' \sin \alpha' )</td>
</tr>
<tr>
<td>( -\sin \gamma' )</td>
<td>( 0 )</td>
<td>( \cos \gamma' )</td>
</tr>
</tbody>
</table>

(24)

Now, let the coordinates of \( F \) referred to \( (x_3, y_3, z_3) \) system be

\[
x_{3F} = OF \sin \theta \cos \alpha_2, \quad y_{3F} = OF \sin \theta \sin \alpha_2, \quad z_{3F} = OF \cos \theta
\] (25)

(see fig. 5)

Fig. 5 Relation of \( x_3' y_3' z_3' \) system to \( x_3 y_3 z_3 \) system and definition of the angles \( \alpha_2, \theta_3, \theta_4 \).

Further, let \( x_3' y_3' z_3' \) be the parallel translation of \( x_3 y_3 z_3 \) system by the translation of \( O \) to \( T \) (see fig. 6), and the coordinates of \( F \) referred to this new system be

\[
x_{3F'} = TF \sin \theta \cos \alpha_2, \quad y_{3F'} = TF \sin \theta \sin \alpha_2, \quad z_{3F'} = TF \cos \theta
\] (26)

So, if \( \theta, \alpha_2 \) are given and \( TF \) are calculated, the position of \( F \) are determined, and in the same way as in \( T \), its height \( OF \) can be obtained. From this and

\[
\sin \theta_4 = \frac{F}{O} \sin \theta
\] (27)

we can get \( \theta_4 \) and so \( x_{3F'} y_{3F'} z_{3F'} \), i.e. the coordinate of \( F \) referring to \( (x_3, y_3 \)
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However the coordinates of T referred to x₄, y₄, z₄ system are
\[x₄T = y₄T = z₄T = 0\]  \hspace{1cm} (28)

By giving these values we are get the coordinates of F, T referred to X, Y, Z system X₁, Y₁, Z₁, X₄, Y₄, Z₄ and so
\[X₄' = X₄, \ Y₄' = Y₄, \ Z₄' = Z₄ - a₀, \ X₄T = X₄T, \ Y₄T = Y₄T, \ Z₄T = Z₄T - a₀\]  \hspace{1cm} (29)
can be determined.

Therefore, the coordinates of T referred to X₁, Y₁, Z₁ system can be obtained as follows:
\[r₁ = (X₄T - X₄') \cosh + (Z₄T - Z₄') \sinh = (X₄ - X₄') \cosh + (Z₄ - Z₄') \sinh\]  \hspace{1cm} (30)
\[\delta₁ = Y₄T' - Y₄' = Y₄T - Y₄\]
\[κ₁ = (Z₄T' - Z₄') \cosh - (X₄T' - X₄') \sinh = (Z₄ - Z₄') \cosh - (X₄ - X₄') \sinh\]

Here, h is the Sun's altitude and the relation between (X₁, Y₁, Z₁) system and (X₄', Y₄', Z₄') system is

\[
\begin{bmatrix}
X'
Y'
Z'
\end{bmatrix}
= \begin{bmatrix}
\cosh & 0 & \sinh \\
0 & 1 & 0 \\
-sinh & 0 & \cosh
\end{bmatrix}
\begin{bmatrix}
X₁
Y₁
Z₁
\end{bmatrix}
\]  \hspace{1cm} (31)

The direction cosines of FT referred to X₁, Y₁, Z₁ axis are
\[\cos ω₁'' = \frac{Y₁}{FT}, \ \cos ω₁' = \frac{δ₁}{FT}, \ \cos ω₁ = \frac{κ₁}{FT}\]  \hspace{1cm} (32)
in which ω₁'', ω₁', ω₁ are angle between FT and X₁ axis.

r₁, δ₁, κ₁ are linear functions of x₄T - x₄F, y₄T - y₄F, z₄T - z₄F by (17), (24) and (30), and they are expressed as follows by (25)
\[x₄T - x₄F = x₄' - x₄'F = -x₄'F = -TF \sin θ₃ \cos A₂\]
\[y₄T - y₄F = y₄' - y₄'F = -y₄'F = -TF \sin θ₃ \sin A₂\]
\[z₄T - z₄F = 0T - OF \cos θ₃ = TF \cos θ₃\]  \hspace{1cm} (35)

by (24)
\[X₄T - X₄F = \cos τ' \cos θ₃ \cos A₂(x₃T - x₃F) - \sin θ₃ \sin A₂(y₃T - y₃F) + \sin τ' \cos θ₃ \cos A₂(z₃T - z₃F)\]
\[Y₄T - Y₄F = \cos τ' \sin θ₃(x₃T - x₃F) + \cos θ₃ \cos A₂(y₃T - y₃F) + \sin τ' \sin θ₃ \cos A₂(z₃T - z₃F)\]
\[Z₄T - Z₄F = - \sin τ' (x₃T - x₃F) + \cos τ' (z₃T - z₃F)\]  \hspace{1cm} (34)

Therefore, cos ω₁', cos ω₁'', cos ω₃, can be evaluated by sine and cosine of τ' A₂, A₂. The coordinates of E referred to X₁', Y₁', Z₁' system are
\[X₁'E = (X₁'E - X₁F) \cosh + (Z₁'E - Z₁F) \sinh\]
\[Y₁'E = Y₁'E' - Y₁F'\]
\[Z₁'E = (Z₁'E - Z₁F) \cosh - (X₁'E - X₁F) \sinh\]  \hspace{1cm} (35)

Then
\[X₁E - δ₁ = (X₁'E - X₁F) \cosh + (Z₁'E - Z₁F) \sinh = (X₁E - X₁T) \cosh + (Z₁E - Z₁T) \sinh\]
\[ Y_{1E} - \delta_1 = Y_{1E} - Y_T' = Y_{E} - Y_T \quad (36) \]

\[ Z_{1E} - \kappa_1 = (Z_{E} - Z_{T}) \cosh - (X_{E}' - X_{T}') \sinh = (Z_{E} - Z_T) \cosh - (X_{E} - X_T) \sinh \]

The direction cosines of \( x_2 y_2 z_2 \) axes with respect to \( X Y Z \) system given in (18) shall be written for brevity by \( a_1, a_2, a_3 b_1, b_2, b_3, c_1, c_2, c_3 \), then

\[
\begin{align*}
X_{E} - X_T &= a_1(x_{2E}' - x_{2T}') + a_2(y_{2E}' - y_{2T}') + a_3(z_{2E}' - z_{2T}') \\
Y_{E} - Y_T &= b_1(x_{2E}' - x_{2T}') + b_2(y_{2E}' - y_{2T}') + b_3(z_{2E}' - z_{2T}') \\
Z_{E} - Z_T &= c_1(x_{2E}' - x_{2T}') + c_2(y_{2E}' - y_{2T}') + c_3(z_{2E}' - z_{2T}')
\end{align*}
\]

(37)

here \( x_{2E}', y_{2E}', z_{2E}' \) being zero by the definition of \( (x_2', y_2', z_2') \) system. Eventually, we get by (18) and (21)

\[
\begin{align*}
X_{E} - X_T &= -\cos r \cos \theta \cos \phi + \sin \theta \sin \phi \cos \theta + \sin r \cos \phi \cos \theta, \\
Y_{E} - Y_T &= -\cos r \sin \theta \cos \phi + \sin \theta \cos \phi \sin \phi + \sin r \sin \phi \sin \theta, \\
Z_{E} - Z_T &= \sin r \sin \phi \cos \theta, + \cos r \cos \theta, \quad (38)
\end{align*}
\]

From this, the direction cosines of \( TE \) line referred to \( x, y, z \) system i.e.

\[
\begin{align*}
\frac{X_{1E} - \tau_1}{TE} &= \frac{Y_{1E} - \delta_1}{TE} = \frac{Z_{1E} - \kappa_1}{TE}
\end{align*}
\]

can be evaluated from (36) by sine and cosine of \( \tau, \theta, \phi \). For abbreviation we will write them by \( l, m, n \).

\[
\begin{align*}
l &= \frac{X_{1E} - \tau_1}{TE}, \quad m = \frac{Y_{1E} - \delta_1}{TE}, \quad n = \frac{Z_{1E} - \kappa_1}{TE}, \quad (39)
\end{align*}
\]

Further, let the direction cosines of \( Z_2 \) and \( Z_2' \) axes referred to \( X_1, Y_1, Z_1 \) system be \( l_2, m_2, n_2 \) and \( l_2', m_2', n_2' \), so we have from (32) and (4)

\[
\begin{align*}
\frac{\tau_1 - \kappa_1}{R_1} &= \cos \omega_1 \cos \omega_1, = l_2 \sin \omega_1 = L_1, \\
\frac{\delta_1 - \kappa_1}{R_1} &= \cos \omega_1 \sin \omega_1, = m_2 \sin \omega_1 = M_1, \\
-\frac{\tau_1 + \beta_1}{R_1} &= \cos^2 \omega_1, = -1 + \cos^2 \omega_1, = n_2 \sin \omega_1 = N_1 \quad (40)
\end{align*}
\]

These three quantities shall be notated by \( L_1, M_1 \) and \( N_1 \).

\[ \cos \omega_2 = l_1 + m_1 n_1 + n_1 \quad (41) \]

And, from (32) and (6) we get

\[
\begin{align*}
\frac{\tau_1 - \gamma_1}{R_1} &= \cos \omega_1, = l_2 \sin \omega_1, = L_2, \\
-\frac{\tau_1 + \beta_1}{R_1} &= \cos^2 \omega_1, = -1 + \cos^2 \omega_1, = m_2 \sin \omega_1, = M_2, \quad (42)
\end{align*}
\]

\[ \frac{\delta_1 - \kappa_1}{R_1} = \cos \omega_1, \cos \omega_1, = n_2 \sin \omega_1, = N_2. \]

Three quantities in (42) shall be notated by \( L_2, M_2 \) and \( N_2 \).
The Phase function in the Tertiary Scattering (1)

As \( \omega_2' \) is the angle between TE and \( Z_2' \)

\[
\cos \omega_2' = l_2' + m_2' + n_2'.
\] (43)

From (40) and (41)

\[
\cos \omega_2 \sin \omega_2' = l_1 + m_1 + n_1, \quad \sin^2 \omega_2 \sin^2 \omega_2' = \sin^2 \omega_2' - (l_1 + m_1 + n_1)^2 = \vartheta.
\] (44) (45)

The expression of (45) shall be for brevity written by \( \vartheta \).

In the same way

\[
\sin^2 \omega, \sin^2 \omega_2' = \sin^2 \omega_2' - (l_1 + m_1 + n_1)^2 = \vartheta'
\] (46)

The direction cosines of \( X_2 \) axis referred to \( X, Y, Z \) system are

\[
-\frac{\tau_1}{R}, \quad -\frac{\delta_1}{R}, \quad -\frac{\kappa_1}{R},
\] (47)

and that of \( Z_2 \) axis \( l_2 m_2 n_2 \), then that of \( Y_2 \) can be determined. Let them be \( \lambda, \mu, \nu \), then we get

\[
\begin{array}{c|c|c|c}
\hline
& X_1 & Y_1 & Z_1 \\
\hline
Xa & \frac{\tau_1}{R} & \lambda & l_2 \\
Y_2 & \frac{\delta_1}{R} & \mu & m_2 \\
Z_2 & \frac{\kappa_1}{R} & \nu & n_2 \\
\hline
\end{array}
\] (48)

The result of computation is as follows

\[
\lambda = \pm \frac{\delta_1}{\sin \omega}, \quad \frac{1}{R}, \quad \mu = \pm \frac{\tau_1}{\sin \omega}, \quad \frac{1}{R}, \quad \nu = 0
\] (49)

We will explain this result.

It is clear that

\[
\lambda \tau_1 + \mu \delta_1 + \nu \kappa_1 = 0
\] (50)

\[
\lambda l_2 + \mu m_2 + \nu n_2 = 0
\] (51)

From (50) \( \times l_2 \rightarrow (51) \times \tau_1 \), we get

\[
\mu (\delta_1 l_2 - \tau_1 m_2) + \nu (\kappa_1 l_2 - \tau_1 n_2) = 0
\] (52)

From (50) \( \times m_2 \rightarrow (51) \times \tau_1 \), we get

\[
\lambda (\tau_1 m_2 - \delta_1 l_2) + \nu (\kappa_1 m_2 - \delta_1 n_2) = 0
\] (53)

From (50) \( \times n_2 \rightarrow (51) \times \kappa_1 \), we get

\[
\lambda (\tau_1 n_2 - \kappa_1 l_2) + \mu (\delta_1 n_2 - \kappa_1 m_2) = 0
\] (54)

From (52), (53) and (54), we get

\[
\frac{\lambda}{\delta_1} \frac{\kappa_1}{\kappa_1} \frac{\mu}{\tau_1} = \frac{\nu}{\tau_1} \frac{\nu}{\delta_1} = \frac{k}{R},
\] (55)
Let us define $k$ by (55). From (40) we get
\[
\delta, n_2 - \kappa, m_2 = -\frac{1}{R\sin \omega_1} \left( \delta, (\tau + \delta) + \kappa \right), \quad 1 = -\frac{\delta_1}{\sin \omega_1},
\]
(56)
\[
\kappa, l_2 - n_2 \tau_1 = -\frac{1}{R\sin \omega_1} \left( \tau, \kappa + \tau_1 (\tau + \delta) \right) = \frac{\tau_1}{\sin \omega_1},
\]
(57)
\[
\tau, m_2 - l_2 \delta_1 = -\frac{1}{R\sin \omega_1} \left( \tau, \delta, \kappa - \tau, \kappa, \delta \right) = 0
\]
(58)
Substitute (55), (56), (57), (58) in
\[
\lambda^2 + \mu^2 + \nu^2 = 1
\]
(59)
Then we get
\[
\frac{\kappa^2}{R\sin \omega_1} \left( \tau + \delta \right) = 1
\]
(60)
When we substitute the third expression of (40) in (60), we get
\[
\kappa^2 = 1
\]
(61)
Hence we have (49) from (55), (56), (57) and (58).

We may use either the upper or the lower sign in (61) as in explained in later.

Let the direction cosines of $Y_2'$ axis and $Z_2'$ axis referring to $(X, Y, Z)$ system be $\lambda', \mu', \nu'$, $l_2', m_2', n_2'$. We have then (62).

<table>
<thead>
<tr>
<th>$X_2'$</th>
<th>$Y_2'$</th>
<th>$Z_2'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>$-\frac{\tau_1}{R_1}$</td>
<td>$\lambda'$</td>
</tr>
<tr>
<td>$Y_1$</td>
<td>$-\frac{\delta_1}{R_1}$</td>
<td>$\mu'$</td>
</tr>
<tr>
<td>$Z_1$</td>
<td>$-\frac{\kappa_1}{R_1}$</td>
<td>$\nu'$</td>
</tr>
</tbody>
</table>

(62)

Therefore, in the same way as in the preceding discussion
\[
\begin{vmatrix}
\delta_1 & \kappa_1 \\
m_2' & n_2'
\end{vmatrix}
\begin{vmatrix}
\kappa_1 & \tau_1 \\
\tau_1 & \delta_1
\end{vmatrix}
= -\frac{k'}{R_1}
\]
(63)
And from (42) we get
\[
\delta, n_2' - \kappa, m_2' = -\frac{1}{R\sin \omega_1} \left( \delta, (\tau + \delta) + \kappa \right), \quad 1 = -\frac{\delta_1}{\sin \omega_1},
\]
(64)
\[
\kappa, l_2' - \tau, n_2' = -\frac{1}{R\sin \omega_1} \left( \kappa, \tau, \delta, \kappa - \tau, \delta, \kappa \right) = 0
\]
(65)
\[
\tau, m_2' - \delta, l_2' = -\frac{1}{R\sin \omega_1} \left( \tau, (\tau + \kappa) - \tau, \delta \right) = -\frac{\tau_1}{\sin \omega_1}
\]
(66)
Substitute (65)–(66) in (67), then we have (68)

\[ \lambda'^2 + \mu'^2 + \nu'^2 = 1 \]  
\[ \frac{k'^2}{R_1} \cdot \frac{1}{\sin^2 \omega_I} = 1 \]  
\[ k'^2 = 1 \]  
\[ k' = \pm 1 \]

Eventually

\[ \lambda' = \pm \frac{1}{R_1} \cdot \frac{k}{\sin \omega_I} \]  
\[ \mu' = 0 \]  
\[ \nu' = \pm \frac{1}{R_1} \cdot \frac{\kappa}{\sin \omega_I} \]  

Here we can reasonably use each of upper and lower signs.

From (48) we get

\[ X_1 = r - \frac{r'}{R_1} X_2 + \lambda Y_2 + l_2 Z_2 \]  
\[ Y_1 = \delta - \frac{\delta'}{R_1} X_2 + \mu Y_2 + m_2 Z_2 \]  
\[ Z_1 = \kappa - \frac{\kappa'}{R_1} X_2 + \nu Y_2 + n_2 Z_2 \]  

And from (62)

\[ X_1 = r - \frac{r'}{R_1} X_2' + \lambda' Y_2' + l_2' Z_2' \]  
\[ Y_1 = \delta - \frac{\delta'}{R_1} X_2' + \mu' Y_2' + m_2' Z_2' \]  
\[ Z_1 = \kappa - \frac{\kappa'}{R_1} X_2' + \nu' Y_2' + n_2' Z_2' \]  

Let the direction cosine of \( Z_3 \) axis referring to \((X_2 Y_2 Z_2)\) system be notated by \((l_3 m_3 n_3)\), then we have

\[ l_3 = \frac{r}{R \sin \omega}, \quad m_3 = \frac{\delta}{R \sin \omega}, \quad n_3 = -\frac{\kappa}{R \sin \omega} \]

Here \((r \delta \kappa)\) is the coordinate of \(E\) referring to \((X_2 Y_2 Z_2)\) system and can be obtained as follows:

\[ r = \frac{r}{R \sin \omega}, \quad \delta = \frac{\delta}{R \sin \omega}, \quad \kappa = \frac{\kappa}{R \sin \omega} \]

Hence we can calculate the above mentioned direction cosines \((l_3 m_3 n_3)\).

Let the direction cosine of \( Z_3 \) axis referring to \((X, Y, Z_1)\) system be \((\xi \eta \zeta)\), then we have (74) from (48)
\[
\xi = -\frac{r_1 l_3}{R_1} + \lambda m_3 + l_2 n_3 \\
\eta = -\frac{\delta_1 l_3}{R_1} + \mu m_3 + m_2 n_3 \\
\zeta = -\frac{\kappa_1 l_3}{R_1} + \nu m_3 + n_2 n_3
\]

(74)

However, from (73), (49) and (40)

\[
\frac{r_2}{R_2} = -\frac{R_1}{\kappa_1} \left( \frac{r_1 \kappa_1}{R_1^2} + m \frac{\delta_1 \kappa_1}{R_1^2} + n \frac{\kappa_1}{R_1^2} \right) = -\frac{R_1}{\kappa_1} (\Sigma l_1 + m M_3 + n (N_3 + 1)) = -\frac{1}{\cos \omega_1} (\Sigma l_1 + n)
\]

(75)

\[
\frac{\delta_2}{R_2} = 1 \lambda + m \mu
\]

(76)

\[
\frac{\kappa_2}{R_2} = l l_2 + m m_2 + n n_2 = \Sigma l l_3
\]

(77)

Then when we define \( L_3, \ M_3, \ N_3 \) by (78)

\[
L_3 = \frac{r_2 \kappa_2}{R_2^2}
\]

\[
M_3 = \frac{\delta_2 \kappa_2}{R_2^2}
\]

(78)

\[
N_3 = -\frac{\kappa_2}{R_2^2}
\]

we have (79) from (72), (75), (76), (77)

\[
L_3 = l_3 \sin \omega_2 = -\frac{1}{\sin \omega_1} \cdot \left( \frac{\Sigma \Sigma l_1}{\cos \omega_1} \right)
\]

\[
M_3 = m_3 \sin \omega_2 = (1 \lambda + m \mu) \frac{\Sigma l l_3}{\sin \omega_1}
\]

(79)

\[
N_3 = n_3 \sin \omega_2 = -\left( 1 - \frac{\kappa_2}{R_2^2} \right) = -\sin^2 \omega_2 = -\frac{\vartheta}{\sin \omega_1}
\]

Substitute (79) in (74) we get

\[
\xi = -\frac{1}{\sin \omega_2} \left( \frac{r_1 l_3}{R_1} + \lambda M_3 + l_2 N_3 \right)
\]

\[
\eta = -\frac{1}{\sin \omega_2} \left( \frac{\delta_1 l_3}{R_1} + \mu M_3 + m_2 N_3 \right)
\]

(80)

\[
\zeta = -\frac{1}{\sin \omega_2} \left( \frac{\kappa_1 l_3}{R_1} + \nu M_3 + n_2 N_3 \right)
\]

By further deformation

\[
\xi = \frac{1}{\sin \omega_1, \sin \omega_2} \left[ \left\{ \frac{r_1 l_3}{R_1} \frac{\Sigma l l_3 + n}{\cos \omega_1} + \lambda (1 \lambda + m \mu) \right\} \left( \Sigma l l_1 \right) - \frac{L_1}{\sin^2 \omega_1, \vartheta} \right]
\]
The Phase function in the Tertiary Scattering (1)

\[ \eta = \frac{1}{\sin \omega_1 \sin \omega_2} \left[ \left\{ \frac{\delta_1}{R_1} \sum I_L + \frac{n}{\cos \omega_1} + \mu (1 + m \mu) \right\} (\sum I_L) - \frac{M_1}{\sin^2 \omega_1 \varphi} \right] \]

\[ \zeta = \frac{1}{\sin \omega_1 \sin \omega_2} \left[ \left\{ \frac{\epsilon_1}{R_1} \sum I_L + \frac{n}{\cos \omega_1} \right\} (\sum I_L) - \frac{N_1}{\sin^2 \omega_1 \varphi} \right] \]

(81)

Represent the large bracket in (81) by \( \xi, \eta, \zeta \), then we have

\[ \xi = \xi \sin \omega_1 \sin \omega_2 \]

\[ \eta = \eta \sin \omega_1 \sin \omega_2 \]

\[ \zeta = \zeta \sin \omega_1 \sin \omega_2 \]

As \((\xi, \eta, \zeta)\) is the directioncosine of \(Z_3\) axis referring to \((X_1, Y_1, Z_1)\) system, the value referring to \((X'Y'Z')\) system reduces to (85) by use of (31)

\[ \xi \cosh - \eta \sinh - \zeta \cosh \]

(83)

The directioncosine of \(EO'\) referring to \((X'Y'Z')\) system is

\[-\cos \theta \cos \Lambda, -\cos \theta \sin \Lambda, -\sin \theta \]

(84)

Thus, as \(\omega_3\) is the angle between \(EO'\) and \(Z_3\) axis, it becomes

\[ \cos \omega = -\cos \theta \cos \Lambda (\xi \cosh - \eta \sinh) - \cos \theta \sin \Lambda \cdot \eta \]

\[-\sin \omega (\xi \sinh + \eta \cosh) \]

(85)

\[ -\sin \omega_1 \sin \omega_2 \cos \omega_3 = \xi, (\sin \theta \sinh + \cos \theta \cosh \cos \Lambda) + \eta, \cos \theta \sinh \cos \Lambda = H \]

(86)

If we denote \(H\) as the right hand side of (86), then

\[ (1 - \sin^2 \omega_3) \sin^2 \omega \sin^2 \omega_2 = H^2 \]

(87)

and when we put

\[ D = \sin^2 \omega_1 \sin^2 \omega_2 \sin^2 \omega_3 \]

(88)

then we get from (45)

\[ D = \eta - H^2 \]

When we denote the directioncosine of \(Z_3'\) axis referring to \((X_2', Y_2', Z_2')\) system as \((l_{3'}, m_{3'}, n_{3'})\), then

\[ l_{3'} = \frac{-r_{3'} \cdot \nu_2'}{R \sin \omega_2}, \quad m_{3'} = \frac{-r_{3'} \cdot \nu_2'}{R \sin \omega_2}, \quad n_{3'} = \frac{-r_{3'} \cdot \nu_2'}{R \sin \omega_2} \]

(89)

Here, \((r_{3'}, \nu_{2'}, x_{2'})\) is the coordinate of \(E\) referring to \((X_2', Y_2', Z_2')\) system and is given as follows by (62)

\[ r_{2'} = -(X_1, x - r_1) \frac{R_1}{R}, \quad \nu_{2'} = -(Y_1, x - r_1) \frac{R_1}{R}, \quad l_{2'} = -(Z_1, x - r_1) \frac{R_1}{R} \]

\[ \delta_{2'} = (X_1, x - r_1) \lambda' + (Y_1, x - r_1) \mu' + (Z_1, x - r_1) \nu' \]

\[ \kappa_{2'} = (X_1, x - r_1) \lambda' + (Y_1, x - r_1) \mu' + (Z_1, x - r_1) \nu' \]

(90)

We can then compute the directioncosine \((l_{3'}, m_{3'}, n_{3'})\) above mentioned. Then the directioncosine of \(Z_3'\) axis referring to \((X_1, Y_1, Z_1)\) system can also be computed from (62) as follows

\[ \xi' = -\frac{r_1}{R_1} l_{3'} + \lambda' m_{3'} + l_{3'} n_{3'} \]
\[
\eta' = -\frac{\delta_1}{R_1} I_3 + \mu' m_3' + m_2' n_3' \quad \text{(91)}
\]

\[
\zeta' = -\frac{\kappa_1}{R_1} I_3 + \nu' m_3' + n_2' n_3' \quad \text{(92)}
\]

Let us denote the value as \((\xi', \eta', \zeta')\). From (39), (42), (43), (69) and (90) we get

\[
\frac{r'_{z_2}}{R_2} = \frac{r_{z_2}}{R_2} = -\frac{1}{\cos \omega_1} (\Sigma I_{l_1} + n) \quad \text{(93)}
\]

\[
\frac{\delta'_{z_2}}{R_2} = \delta_{z_2} = 1 + m \mu' n \nu' = 1 + n \nu' \quad \text{(94)}
\]

\[
\frac{\kappa'_{z_2}}{R_2} = \kappa_{z_2} = m m_2' + n n_2' = \frac{\Sigma I_{l_2}}{\sin \omega_1} = \cos \omega_{z_2}' \quad \text{(95)}
\]

Then, when we put

\[
L_{3} = -\frac{r'_{z_2} \kappa'_{z_2}}{R_2^2}
\]

\[
M_{3}' = -\frac{\delta'_{z_2} \kappa'_{z_2}}{R_2^2}, \quad N_{3}' = -\frac{\delta'_{z_2} \delta_{z_2}'}{R_2^2} \quad \text{(96)}
\]

we get from (89), (92), (93), (94), and (46)

\[
L_{3} = \sin \omega_{z_2}' = -\frac{\Sigma I_{l_1} + n}{\cos \omega_1}, \quad \frac{\Sigma I_{l_2}}{\sin \omega_1}
\]

\[
M_{3}' = m_{3}' \sin \omega_{z_2}' = (1 + n \nu') \frac{\Sigma I_{l_2}}{\sin \omega_1}
\]

\[
N_{3}' = n_{3}' \sin \omega_{z_2}' = (1 - \frac{\kappa_{z_2}}{R_1}) = -\frac{\sin^2 \omega_{z_2}'}{-\sin^2 \omega_1}
\]

Substitute (96) in (91), then we get

\[
\xi' = -\frac{r_{z_2} - L_{3} + \lambda M_{3}' + 1 N_{3}'}{\sin \omega_{z_2}'}
\]

\[
\eta' = -\frac{\delta_{z_2} - L_{3} + \mu' M_{3}' + m_{2}' N_{3}'}{\sin \omega_{z_2}'} \quad \text{(97)}
\]

\[
\zeta' = -\frac{\kappa_{z_2} + L_{3} + \nu' M_{3} + n_{2}' N_{3}'}{\sin \omega_{z_2}'}
\]

By further transformation by (42) and (96)

\[
\xi' = \left[ \frac{\Sigma I_{l_1} + n \mu}{\cos \omega_1} \right] \left( \Sigma I_{l_2} - \frac{L_{z_2}}{\sin^2 \omega_1} \right) / \sin \omega_1, \sin \omega_{z_2}'
\]

\[
\eta' = \left[ \frac{\delta_{z_2} - L_{3} + \mu' M_{3}' + m_{2}' N_{3}'}{\cos \omega_1} \right] \left( \Sigma I_{l_2} - \frac{M_{z_2}}{\sin^2 \omega_1} \right) / \sin \omega_1, \sin \omega_{z_2}'
\]

\[
\zeta' = \left[ \left( \Sigma I_{l_1} + n \mu' \right) - \nu' \left( \Sigma I_{l_2} - \frac{N_{z_2}}{\sin^2 \omega_1} \theta' \right) / \sin \omega_1, \sin \omega_{z_2}'
\]

\[
\eta' = \left[ \frac{\delta_{z_2} - L_{3} + \mu' M_{3}' + m_{2}' N_{3}'}{\cos \omega_1} \right] \left( \Sigma I_{l_2} - \frac{M_{z_2}}{\sin^2 \omega_1} \right) / \sin \omega_1, \sin \omega_{z_2}'
\]

\[
\zeta' = \left[ \left( \Sigma I_{l_1} + n \mu' \right) - \nu' \left( \Sigma I_{l_2} - \frac{N_{z_2}}{\sin^2 \omega_1} \theta' \right) / \sin \omega_1, \sin \omega_{z_2}'
\]
When we denote the large blacket in (98) by \( \xi', \eta', \zeta' \), then we have
\[
\begin{align*}
\xi' &= \xi \sin \omega' \sin \omega_z' \\
\eta' &= \eta \sin \omega' \sin \omega_z' \\
\zeta' &= \zeta \sin \omega' \sin \omega_z' 
\end{align*}
\]
(99)

As \( (\xi', \eta', \zeta') \) is the direction cosine of \( Z_s' \) axis referring to \( (X, Y, Z_1) \) system, the value referring to \( (X', Y', Z) \) system is given by (100) from (31).
\[ \xi' \cos h - \zeta' \sin h, \quad \eta', \quad \xi' \sin h + \zeta' \cos h \]
(100)

As \( \omega_s' \) is the angle between \( EO' \) and \( Z_s' \) axis, then we have from (100) and (84)
\[
\begin{align*}
\cos \omega_s' &= -\cos \theta \cos A (\xi' \cos h - \zeta' \sin h) - \cos \theta \sin A \cdot \eta' \\
&\quad - \sin \theta (\xi' \sin h + \zeta' \cos h) \\
&\quad - \sin \omega' \sin \omega_z' \cos \omega_s' \allowbreak 
= (\sin \theta \sin h + \cos \theta \cos h \cos A) \xi' \\
&\quad + \cos \theta \sin A \cdot \eta' \\
&\quad + (\sin \theta \cos h - \cos \theta \sin h \cos A) \zeta' \\
&= H'
\end{align*}
\]
(102)

We will denote the right hand side of (102) by \( H' \), then
\[ (1 - \sin^2 \omega_s') \sin^2 \omega_s' = H'^2 \]
(103)

and when we put
\[ D' = \sin \omega_s' \sin^2 \omega_s' \sin^2 \omega_z' \]
then we get from (46)
\[ D' = \vartheta' - H'^2 \]
(104)

From the above detailed explanation we can evaluate the blacket of (14).

We must now proceed to the discussion of the sign of \( (\lambda' \mu' \nu') \) in (49). If we adopt the lower sign of (49), it is clear from (73) that the sign of \( \delta_2 \) will change, therefore the sign of \( m_3 \) will also change from (72), the value of \( (\xi, \eta, \zeta) \) will be invariable from (72), so that \( (\xi, \eta, \zeta) \) will also be invariable from (81), then \( \cos \omega_s' \) is also so from (85). Moreover \( H \) and \( D \) are invariable from (86) and (88).

In the same way as in this discussion, \( D' \) is invariable either when we adopt the upper sign or the lower sign of (69).

4. Particular case. We can get \( \theta_2 \) by giving the right hand side of (20), then \( r' \) and \( A' \) will be gained from (23) by giving \( A, A_1, r \). By giving \( \theta_3, A_2 \) we can get \( r_1/R_1, \theta_1/R_1, \kappa_1/R_1 \) from (33), (34) and (30).

We can get \( 1, m, n, \) from (38) and (36), and \( L, M, N, \) \( l_2m_2n_2 \) from (40), \( \omega_2 \) from (41), \( L_2M_2N_2 \) \( l_2'm_2'n_2' \) from (42), \( \omega_2' \) from (45), \( \theta \) from (45) \( \vartheta' \) from (46), \( (\lambda', \mu', \nu') \) from (49), \( (\xi, \eta, \zeta,') \) from (81), \( H \) from (86) and eventually \( D \) from (88).

In the same way, we can get \( (\lambda' \mu' \nu') \) from (69), \( (\xi', \eta', \zeta,') \) from (98), \( H' \) from (102) and at last \( D' \) from (104).

To simplify the discussion, let us put
\[
\frac{X_T - X_F}{T_F} = a, \quad \frac{Y_T - Y_F}{T_F} = b, \quad \frac{Z_T - Z_F}{T_F} = c
\] (105)

then we have from (33) and (34)

a) The case \( A_2 = 0 \)
\[
a = \cos A \sin (\tau' - \theta_2) \\
b = \sin A \sin (\tau' - \theta_2) \\
c = \cos (\tau' - \theta_2)
\] (106)

b) The case \( A_2 = \frac{\pi}{2} \)
\[
a = -\sin A \sin \theta_2 + \sin \tau' \cos A \cos \theta_2 \\
b = -\cos A \sin \theta_2 + \sin \tau' \sin A \cos \theta_2 \\
c = \cos \tau' \cos \theta_2
\] (107)

c) The case \( A_2 = \pi \)
\[
a = \cos A \sin (\tau' + \theta_2) \\
b = \sin A \sin (\tau' + \theta_2) \\
c = \cos (\tau' + \theta_2)
\] (108)

d) The case \( A_2 = \frac{3\pi}{2} \)
\[
a = \sin A \sin \theta_2 + \sin \tau' \cos A \cos \theta_2 \\
b = \cos A \sin \theta_2 + \sin \tau' \sin A \cos \theta_2 \\
c = \cos \tau' \cos \theta_2
\] (109)

We have further from (23)

I) The case \( A = 0 \)
\[
\sin \tau' \cos A' = \cos \tau \sin \theta_2 \cos A + \sin \tau \cos \theta_2 \\
\sin \tau' \sin A' = \sin \theta_2 \sin A \\
\cos \tau' = -\sin \tau \sin \theta_2 \cos A + \cos \tau \cos \theta_2
\] (110)

1) The case \( A_1 = 0 \)
\[
\sin \tau' \cos A' = \sin (\tau + \theta_2) \\
\sin \tau' \sin A' = 0 \\
\cos \tau' = \cos (\tau + \theta_2)
\] (111)

then we get form (106)~(109)

a) \( a = \sin (\tau + \theta_2 - \theta_3) \)
\( b = 0 \)
\( c = \cos (\tau + \theta_2 - \theta_3) \) (112)

And we get from (30)
\[
\begin{align*}
\frac{r}{R} & = \sin (\tau + \theta_2 - \theta_3 + h) \\
\frac{\delta}{R} & = 0 \\
\frac{\kappa}{R} & = \cos (\tau + \theta_2 - \theta_3 + h)
\end{align*}
\] (113)
b) \[
\begin{align*}
\mathbf{a} &= \sin r' \cos \theta_3 \\
\mathbf{b} &= -\sin \theta_3 \\
\mathbf{c} &= \cos r' \cos \theta_3
\end{align*}
\]

\[
\frac{R_1}{r_1} = \cos \theta_3 \sin (r + \theta_2 + h)
\]

\[
\frac{\delta_1}{R_1} = -\sin \theta_3
\]

\[
\frac{\kappa_1}{R_1} = \cos \theta_3 \cos (r + \theta_2 + h)
\]

c) \[
\begin{align*}
\mathbf{a} &= \sin (r' + \theta_3) \\
\mathbf{b} &= 0 \\
\mathbf{c} &= \cos (r' + \theta_3)
\end{align*}
\]

\[
\frac{R_1}{r_1} = \sin (r + \theta_2 + \theta_3 + h)
\]

\[
\frac{\delta_1}{R_1} = 0
\]

\[
\frac{\kappa_1}{R_1} = \cos (r + \theta_2 + \theta_3 + h)
\]

d) \[
\begin{align*}
\mathbf{a} &= \sin r' \cos \theta_3 \\
\mathbf{b} &= \sin \theta_3 \\
\mathbf{c} &= \cos r' \cos \theta_3
\end{align*}
\]

\[
\frac{R_1}{r_1} = \cos \theta_3 \sin (r + \theta_2 + h)
\]

\[
\frac{\delta_1}{R_1} = \sin \theta_3
\]

\[
\frac{\kappa_1}{R_1} = \cos \theta_3 \cos (r + \theta_2 + h)
\]

In this paper the values of \(D, D'\) in the case I) 1) c) are tabulated in Table 1 for \(r = 0\), here the value of \(\theta, \theta_1, \theta_2, \theta_3\) and \(h\) are expressed in unit of degree. In this case \(D + D' = 1 + \theta \sin^2(\theta - \theta_1)\). We have the values in the case of I) 1) a) when we write the column of \(\theta_3\) upside down i.e. substitute \(\pi - \theta_3\) for \(\theta_3\). At the end the author expresses his sincere thanks to Dr. K. Y. Kondratyev, Rector of Leningrad University, for encouragement and interest to the research.

References

1) L. Rayleigh, 1899 : Phil. Mag., 47
2) M. Planck : Sitz. Ber. Berlin, 470(1902) ; 749(1904)
Table 1. \( \theta = 0 \)

This table expresses the value of the polarization angle in the case of I) 1) c).

<table>
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<tr>
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<th>3</th>
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</tbody>
</table>

This table continues with more values for different \( h \) and \( \theta_1, \theta_2 \) combinations, each representing the polarization angle in various cases.
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</tr>
<tr>
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</tr>
<tr>
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<td>1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624</td>
</tr>
<tr>
<td>150</td>
<td>1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624 1.8624</td>
</tr>
</tbody>
</table>

The phase function in the Tertiary Scattering (1)
<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>30</th>
<th>60</th>
<th>90</th>
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</thead>
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<td>1.0000</td>
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Table 1. $A = 0 \quad \theta = 30$
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<th>2</th>
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<th>2</th>
<th>3</th>
<th>4</th>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
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<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 1. θ = 60

Note: The table entries represent values of a function or a sequence, with θ varying from 0 to 90 degrees. The entries are equal to θ = 0 for θ = 0, and show a progression for increasing θ values.
Table 1. \( A = 0 \quad \theta = 90 \)

| \( h \) | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 3 | 4 |
|------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0    | 1.0000 | 1.0002 | 1.0007 | 1.0014 | 1.0026 | 1.1406 | 1.1462 | 1.1515 | 1.1566 | 1.1612 | 1.4219 | 1.4214 | 1.4207 | 1.4189 | 1.4164 | 1.6251 | 1.5625 | 1.5510 | 1.5387 | 1.5265 | 1.5129 |
| 30   | 1.0469 | 1.0469 | 1.0464 | 1.0458 | 1.0451 | 1.1406 | 1.1549 | 1.1289 | 1.1227 | 1.1165 | 1.8785 | 1.7864 | 1.6540 | 1.5435 | 1.4166 | 1.2951 | 1.1188 | 1.1087 | 1.0990 | 1.0902 | 1.0825 |
| 60   | 1.0000 | 1.0002 | 1.0009 | 1.0017 | 1.0029 | 1.0000 | 1.0009 | 1.0021 | 1.0036 | 1.0059 | 1.0115 | 1.0026 | 1.0000 | 1.0003 | 1.0009 | 1.0015 | 1.0026 | 1.0003 | 1.0002 | 1.0004 | 1.0008 |
| 75   | 1.0469 | 1.0557 | 1.0648 | 1.0685 | 1.0766 | 1.0469 | 1.0526 | 1.0587 | 1.0647 | 1.0713 | 1.0251 | 1.0275 | 1.0298 | 1.0321 | 1.0544 | 1.0534 | 1.0635 | 1.0652 | 1.0651 | 1.0029 |
| 105  | 1.3499 | 1.3586 | 1.3668 | 1.3746 | 1.3821 | 1.1875 | 1.1875 | 1.1856 | 1.1859 | 1.0251 | 1.0227 | 1.0205 | 1.0179 | 1.0156 | 1.0252 | 1.0297 | 1.4344 | 1.0596 | 1.0483 | 1.0035 |
| 120  | 1.4219 | 1.4216 | 1.4205 | 1.4189 | 1.4164 | 1.1406 | 1.1454 | 1.1289 | 1.1227 | 1.1165 | 1.0000 | 1.0002 | 1.0009 | 1.0016 | 1.0050 | 1.1406 | 1.1521 | 1.1642 | 1.1766 | 1.1895 |
| 150  | 1.5175 | 1.1972 | 1.1651 | 1.1541 | 1.1435 | 1.0000 | 1.0009 | 1.0020 | 1.0037 | 1.0047 | 1.0185 | 1.2908 | 1.2104 | 1.2219 | 1.2544 | 1.5625 | 1.5935 | 1.6841 | 1.5988 | 1.6029 |

\( h \) equal to \( \theta = 30 \)

<table>
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<tr>
<th>( h )</th>
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<th>30</th>
<th>60</th>
<th>75</th>
<th>90</th>
<th>105</th>
<th>120</th>
<th>150</th>
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<td>1.0007</td>
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<td>1.0026</td>
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<td>1.1462</td>
<td>1.1515</td>
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</tbody>
</table>

\( h \) equal to \( \theta = 60 \)

<table>
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<th>0</th>
<th>30</th>
<th>60</th>
<th>75</th>
<th>90</th>
<th>105</th>
<th>120</th>
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<td>1.1462</td>
<td>1.1515</td>
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</table>

\( h \) equal to \( \theta = 90 \)

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<th>60</th>
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</table>

The Phase function in the Tertiary Scattering (1)

equal to $\theta = 30$