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On the Ring Satisfying the Finite Continuous Quotient’s Chain of the Ideal

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The object of this report is to find out some qualities of this ring satisfying the finite continuous quotient’s chain of the ideal and to examine a nilpotent element of the ring which has the representation as the intersection of the finite number of strong-primary ideals further.

1. On the ring satisfying the finite continuous quotient’s chain of the ideal

Definition

Set \( \mathfrak{D} \) of all elements which are nilpotent with respect to \( a \) is a semiprimary ideal including \( a \). This \( \mathfrak{D} \) is called semi-primary ideal belonging to \( a \).

Theorem 1.

Let \( \mathfrak{R} \) be a commutative ring satisfying the finite continuous quotient’s chain of the ideals. Then there are the least-primary ideals containing arbitrary ideal \( a \) of \( \mathfrak{R} \) and the ideal \( r \) which satisfy \( p \) which satisfy \( p^a \subseteq a \) for finite positive integer \( n \), moreover, either \( r \) coincides with \( p \) or \( r \) is not contained in \( p \).

Proof

Let \( \mathfrak{D} \) be a semi-primary ideal belonging to \( a \), then we must consider the following two cases.

I. When \( \mathfrak{D} \) is a primary ideal

As \( \mathfrak{D} \) is a least-primary ideal, we can put \( \mathfrak{D} = p \). If \( a \) is not a semi-primary ideal, there are two elements \( h_1, r_1 \) which satisfy the following formulas,

\[
\begin{align*}
    h_1 r_1 & \subseteq a \\
    h_1 & \subseteq p \\
    h_1 & \subseteq a \\
    r_1 & \subseteq p
\end{align*}
\]

hence \( a \subseteq a_1 = (r_1) \) exist. Moreover, \( p \) is a semi-primary ideal belonging to \( a_1 \). If \( a_1 \) is not a semi-primary ideal yet, there are two elements \( h_2, r_2 \), which satisfy the following formulas,

\[
\begin{align*}
    h_2 r_2 & \subseteq a_1 \\
    h_2 & \subseteq p \\
    h_2 & \subseteq a_1 \\
    r_2 & \subseteq p
\end{align*}
\]

and \( a \subseteq a_1 \subseteq a = (r_2) = (r_1 r_2) \subseteq p \) exist.
is held. But from the assumption, this procedure must come to end, therefore \( a_m \) satisfying this formula

\[
(1) \quad a_m = a_1 : (r) \subseteq p \quad r = r_1 r_2 \cdots r_m \subseteq p
\]

becomes semi-primary ideal belonging \( p \) at least. If \( a_m = p \) by letting \( r \) take the place of \( (r) \) we get

\[
p \subseteq a, \quad r \subseteq p
\]

Therefore our theorem is proved. In the next place we examine the following case, \( a_m \subseteq p \). Since \( p \) is a semi-primary ideal belonging to \( a_m \), so \( h_1' \subseteq p \), \( h_1' \subseteq a_m \) is nilpotent with respect to \( a_m \). Moreover, as \( a_m \) is semi-primary ideal, \( a_m \subseteq a_1' = a_m : (h_1') \subseteq p \) exist and \( a_1 \) is a semi-primary ideal belonging to \( a_1' \). If \( a_1' \) does not coincide with \( p \), this procedure is repeated. Hence, we can finally find out an element \( h' \) satisfying \( p = a_1' = a_m : (h') \), \( h' = h_1' h_2' \cdots h_n' \subseteq a_m \), \( h' \subseteq p \) (1)' from the assumption. As \( a_m \) is a semi-primary ideal we have

\[
(2) \quad a_m \subseteq q_1 = a_m : p \subseteq p
\]

from (1'). Also, \( q_1 \) is a semi-primary ideal. From (1) and (2) we have

\[
(3) \quad a \subseteq q_1 = a : (r) \subseteq p : a \quad \text{is not semi-primary}
\]

\[
(3)' \quad a \subseteq q_1 = a : p \quad : a \quad \text{is semi-primary}
\]

As \( q_1 \) is semi-primary ideal belonging to \( p \) here, in the same way as the preceding, we have

\[
(4) \quad q_1 \subseteq q_2 = q_1 : p = a_m : p^2 = a : (r)p^2 \subseteq p, \quad r \subseteq p : a \quad \text{is not semi-primary}
\]

\[
(4)' \quad q_1 \subseteq q_2 = q_1 : p = a : p^2 \quad : a \quad \text{is semi-primary}
\]

From the assumption this procedure must come to end, hence

\[
p = a_{k-1} = a : (r)p^{k-1} \quad r \not\subseteq p \quad \text{or} \quad p = a_{k-1} = a : p^{k-1}
\]

is get at last. Namely, we have \( p \subseteq a, \quad r \not\subseteq p \) and moreover, either \( r \) coincide with \( p \) or \( r \) is not contained in \( p \).

II. When \( \mathfrak{D} \) is not prime ideal

From the assumption there is a element \( r_0 \) satisfying

\[
(5) \quad p = \mathfrak{D} : (r_0) \quad r_0 \subseteq p
\]

This primary ideal \( p \) is clearly a least-primary ideal including \( a \). As \( p r_0 \subseteq \mathfrak{D} \) exist for arbitrary element \( p \) of \( p \) does not belonging to \( \mathfrak{D} \), \( (p r_0) \subseteq a \) is held. Therefore, for \( r_0 r = r_1 \), ideal quotient \( a_1 = a : (r_1) \) implies \( p^* \). But as \( p \) is not belong to \( \mathfrak{D} \), \( p^* \) is not also belong to \( \mathfrak{D} \). Accordingly \( p^* \) does not contained to \( a \) and

\[
(6) \quad a \subseteq a_1 = a : (r_1) \subseteq p, \quad r_1 \subseteq p, \quad p^* \subseteq a,
\]
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is held. For a semi-primary ideal \( \mathfrak{P} \) of \( a \), if \( \mathfrak{P} \subseteq \mathfrak{P} \) hold for a semiprimary ideal \( \mathfrak{P} \) of \( a \) exist from (6), so we have \( \mathfrak{P} \subseteq \mathfrak{P} \subseteq \mathfrak{P} \subseteq \mathfrak{P} \subseteq \mathfrak{P} \subseteq \mathfrak{P} \). If \( \mathfrak{P} \) is a primary ideal, there is an element \( \gamma' \) satisfying \( (\mathfrak{P}, \gamma') \subseteq \mathfrak{P} \). If \( \gamma' \) is not a element of \( \mathfrak{P} \), in the same way as the preceding.

For this element \( \gamma' \), \( (\mathfrak{P}, \gamma') \subseteq \mathfrak{P} \), exist. So, if we take \( (\gamma', \gamma') \subseteq \mathfrak{P} \), is held. Also, \( \mathfrak{P} \) is not contained in \( \mathfrak{P} \), so the semi-primary ideal \( \mathfrak{P} \) of \( a \) implies \( \mathfrak{P} \) and \( \mathfrak{P} \subseteq \mathfrak{P} \subseteq \mathfrak{P} \) is obtained. By rotation of this procedure we get \( \mathfrak{P} \subseteq \mathfrak{P} \subseteq \mathfrak{P} \subseteq \mathfrak{P} \). From the assumption, therefore, we can find an ideal \( \gamma' \) satisfying

\[
(7) \quad a \subseteq a : (\mathfrak{P}, \gamma') \subseteq \mathfrak{P}.
\]

is held. Also, \( \mathfrak{P} \) is not contained in \( \mathfrak{P} \), so the semi-primary ideal \( \mathfrak{P} \) of \( a \) implies \( \mathfrak{P} \) and \( \mathfrak{P} \subseteq \mathfrak{P} \). From the result of 1. From (6) and (7)

\[
\mathfrak{P} = a : (\gamma_1 \gamma_2 \cdots \gamma_k) \subseteq \mathfrak{P}.
\]

is held. Hence, if we put \( \gamma = \gamma_1 \gamma_2 \cdots \gamma_k \), \( \mathfrak{P} \subseteq \mathfrak{P} \) is held for \( \gamma \subseteq \mathfrak{P} \) and \( \mathfrak{P} \subseteq \mathfrak{P} \) for \( \gamma \subseteq \mathfrak{P} \) from (8) and (9).

Namely our theorem is proved.

Definition

\( \mathfrak{P} \) is called primary ideal belonging to \( a \) if there is an element \( d \) satisfying \( \mathfrak{P} = a : (d) \), \( d \subseteq a \) for a primary ideal \( \mathfrak{P} \). As semi-primary ideal \( \mathfrak{Q} \) belonging to semi-primary ideal \( \mathfrak{Q} \) is a primary ideal we get \( \mathfrak{Q} \subseteq \mathfrak{Q} \) from the first case of theorem 1.

Accordingly we have the following theorem.

Theorem 2.

Let \( \mathfrak{R} \) be a commutative ring satisfying the finite continuous quotient's chain of ideals. Then there is a finite positive integer \( n \) satisfying \( \mathfrak{R} \subseteq \mathfrak{R} \) for the semi-primary ideal \( \mathfrak{R} \) belonging ideal \( a \) of \( \mathfrak{R} \).

Proof

It is evident that there are least-primary ideals from theorem 1, so let \( \mathfrak{P}_1, \mathfrak{P}_2, \ldots \) are least-primary ideals. As we can find an ideal \( \mathfrak{P}_i \) satisfying \( \mathfrak{P}_1 \subseteq \mathfrak{P}_i \subseteq a \), \( \mathfrak{P}_1 \subseteq \mathfrak{P}_i \) from theorem 1, \( a \subseteq a : \mathfrak{P}_1, \mathfrak{P}_2, \ldots \) is held. Also \( \mathfrak{P}_1, \mathfrak{P}_2, \ldots \) are least-primary ideals, so we have

\[
a_1 \subseteq a_2 = a_1 : \mathfrak{P}_1, \mathfrak{P}_2, \ldots \subseteq a_1 : \mathfrak{P}_1, \mathfrak{P}_2, \ldots (i=3,4,\ldots)
\]

in the same way as the preceding. But as this procedure must come to an end from the assumption, we have

\[
a_0 = a : \mathfrak{P}_1, \mathfrak{P}_2, \ldots \subseteq a
\]

at last. On the other hand, as \( \mathfrak{P} \subseteq \mathfrak{P}_i \) \((i=1,2,\ldots)\) exist, we get \( \mathfrak{P} \subseteq a \) by putting
2. On nilpotent element in commutative ring generally it is known that in a ring, if it has a nilpotent, then there is a nilpotent ideal, but its inverse does not always exist. Only if we suppose the division chain's condition further, then the ring has a nilpotent element. Here, however, if any ideal of a commutative ring has the representation as the intersection of the finite number strong primary ideals further, then the ring also has a nilpotent element. This is the extension of the former.

**Theorem**

Let \( \mathfrak{A} \) be a commutative ring whose arbitrary ideals have the representation as the intersection of finite number of the strong semi-primary ideals. Then the nilpotent ideal does not being zero of \( a \), implies a nilpotent element outside of zero element.

**Proof**

Let \( a \) be any ideal of a commutative ring, and let \( P_1, P_2, \ldots, P_r \) are the least-primary ideals of \( a \). Suppose that \( q \) is an element of \( a \) does not being zero. Then there exist suitable elements \( P_{i_1}, P_{i_2}, \ldots, P_{i_j} \) for every \( P_i \) and natural number \( n_i \) such that

(1) \( P_i^{n_i} \subseteq \langle P_{i_1}, P_{i_2}, \ldots, P_{i_j}, q \rangle \subseteq a \quad i = 1, 2, \ldots, r \)

Since \( a^2 = a \) and \( a \subseteq P_i \) we see that for any natural number \( s \).

(2) \( a \subseteq \langle P_{i_1}, P_{i_2}, \ldots, P_{i_j}, q \rangle \subseteq P_i^s \quad (i = 1, 2, \ldots, r) \)

is held. Now, from the product of \( r \) number formulas in (2), we obtain

\[ a = a \subseteq \langle P_{i_1}, P_{i_2}, \ldots, P_{i_j}, q \rangle \subseteq \langle P_{i_1}, P_{i_2}, \ldots, P_{i_j}, q \rangle \]

But by the hypothesis we have

\[ a = q_1 \cap q_2 \cap \cdots \cap q_r \]

Where \( q_i \) are the strong primary ideals. Therefore, since

\[ q_1 q_2 \cdots q_r \subseteq a \]

and even \( P_i \) is the least-primary ideals of \( a \), so it follows that for suitably large number \( s \)

\[ P_i^s \subseteq a \]

and therefore

\[ a = \langle P_{i_1}, P_{i_2}, \ldots, P_{i_j}, q \rangle \subseteq \langle P_{i_1}, P_{i_2}, \ldots, P_{i_j}, q \rangle \]

Thus we may write

(3) \[ a = (a_1, a_2, \ldots, a_m) \]
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By (3), we see that for any natural number \( t \)
\[
(a_1^t, a_2^t, \ldots, a_n^t) \geq (a_1, a_2, \ldots, a_n)^t = a^t = a
\]

However, it follows that
\[
a = (a_1, a_2, \ldots, a_m) \geq (a_1^t, a_2^t, \ldots, a_m^t)
\]
hence, there exist the following formula
\[
(4) \ a = (a_1^t, a_2^t, \ldots, a_n^t)
\]
Since \( t \) is arbitrary, we can take \( t \geq 2 \), so
\[
\begin{align*}
a_1 &= a_1a_1^2 + a_2a_2^2 + \cdots + a_na_n^2 \\
\vdots \\
a_m &= a_1a_1^m + a_2a_2^m + \cdots + a_na_n^m
\end{align*}
\]
is held by (3), (4). By multiplying every equivalence of (5) by \( a \in \alpha \), where \( a \neq 0 \), we have
\[
\begin{align*}
(a_{11} - a) a_1 + a_1^2 a_2 + \cdots + a_n a_n &= 0 \\
\vdots \\
(a_{nm} - a) a_m &= 0
\end{align*}
\]
Now, let \( D \) be the determinant of the coefficient \( a_1, a_2, \ldots, a_m \), then \( Da_t = 0 \) and therefore \( D^2 = 0 \) is held. By developing \( D \), we have
\[
a_{m^2} = a' a_{m^2}
\]
Since \( a \) is arbitrary element of \( \alpha \) so we replace \( a \) with \( a_1 \) hence we have
\[
a_{t^m} = a' a_{t^m}
\]
and accordingly \( a^2 = a' \) is get.
Moreover, since \( a' \neq 0 \) and \( aa' = a \) are held for any element \( a \) of \( \alpha \) so \( a' \) is a unite element.
Thus our theorem is proved.

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