On a Property of Schlicht Functions.

Kunio Yamaguchi

A function $w = f(z)$ is said to be schlicht (univalent) in a domain $D$ if for any two points $z_1$ and $z_2$ belonging to $D$ we have $f(z_1) = f(z_2)$ only if $z_1 = z_2$. We shall deal with functions $f(z)$ which are regular and schlicht in the 'unit' circle $|z|<1$, and which are normalized at $z=0$ by $f(0) = 0$, $f'(0) = 1$. Thus, we shall say that $f(z) \subset (S)$ if

$$f(z) = \sum_{n=1}^{\infty} a_n z^n = z + a_2 z^2 + a_3 z^3 + \cdots$$

is regular and schlicht for $|z| < 1$. We shall also say that $f(z) \subset (C)$ if $f(z) \subset (S)$ and more-over

$$|a_n| \leq 1 \ (n=2,3,\ldots).$$

Lemma 1. (Gronwall)

Let the function $f(z) \subset (S)$, then

$$|f(z)| \leq 1 + \frac{|z|}{|a_2| |z| + |z|^2},$$

where $|z| < 1$, with equality holding only for

$$f(z) = \frac{z}{1 - a_2 z + \frac{|z_0|^2}{z_0 z^z}} , \quad \arg a_2 = -\arg z_0 + \pi \ (\text{mod} \ 2\pi)$$

where $z_0$ is any number in $0 < |z_0| < 1$.

Lemma 2. (Goluzin)

Let the function $f(z) \subset (S)$, then

$$\left| \frac{f(z_i) - f(z_j)}{z_i - z_j} \right| \geq |f(z_i) f(z_j)| \frac{1 - r^2}{r^2},$$

where $|z_i| = |z_j| = r, \ 0 < r < 1$, with equality holding only for
\[ f(z) = \frac{z}{1 + ae^{i\theta}z + e^{i\theta}z^*}, \quad \theta = \frac{1}{2}(\arg z_1 + \arg z_2) \]

\[ -2 \leq a \leq 2. \]

From these lemmas we obtain at once two following theorems.

**Theorem 1.**

Let the function \( f(z) \subset (C) \), then

\[ |f'(z)| \geq \frac{1}{1 + |z| + |z|^*} \cdot \frac{1 - |z|}{1 + |z|}, \]

where \( |z| < 1 \).

**Proof.**

By the well-known distortion theorem for \( f(z) \subset (S) \)

\[ \frac{1 - |z|}{1 + |z|} \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \frac{1 + |z|}{1 - |z|}, \]

and by the lemma 1 and the assumption \( |a_i| \leq 1 \)

\[ |f(z)| \geq \frac{|z|}{1 + |z| + |z|^*}, \]

therefore it follows that

\[ |f'(z)| = \left| \frac{f(z)}{z} \cdot \frac{zf'(z)}{f(z)} \right| \geq \frac{1}{1 + |z| + |z|^*} \cdot \frac{1 - |z|}{1 + |z|}, \]

where \( |z| < 1 \).

**Theorem 2.**

Let the function \( f(z) \subset (C) \), then

\[ \left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} \right| \geq \frac{(1 - |z_1|)^2}{1 + |z_1| + |z_2|^2} \cdot \frac{|1 - \bar{z}_1 z_2|}{(|1 - \bar{z}_1 z_2| + |z_2 - z_1|)^2} \]

where \( z_1, z_2 \) are any two points such that \( |z_1| < 1, \ |z_2| < 1 \).

**Proof.**
Applying the theorem 1 to the distortion theorem for \( f'(z) \subset (S)^{(2)} \)

\[
\left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} \right| \geq |f'(z)| \left| \frac{1 - |z_2|^2}{\left(1 - |z_1 z_2| + |z_1 - z_2|\right)^2} \right|, \quad (|z_1| < 1, |z_2| < 1)
\]

we see that

\[
\left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} \right| \geq \frac{1 - |z_2|}{(1 + |z_1| + |z_1|^2)(1 + |z_2| + |z_2|^2)} \cdot \left| \frac{1 - |z_2|^2}{\left(1 - |z_1 z_2| + |z_1 - z_2|\right)^2} \right|, \quad (|z_1| < 1, |z_2| < 1).
\]

**Theorem 3.**

Let the function \( f(z) \subset (C) \), then every section of the power series

\[
\sum_{n=1}^{\infty} a_n z^n \quad (n = 1, 2, 3, \ldots)
\]

is univalent for \( |z| < \frac{1}{2} \) with \( |1 - z| > 49/48 (|z| = 1, z \neq 1) \), except for the cases \( n = 2, 3 \).

**Proof.**

From the lemma 1, 2, we derive

\[
\left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} \right| \geq \left| \frac{1 - |z_2|}{1 + |z_1| + |z_1|^2} \right| \left| \frac{1 - |z_2|^2}{\left(1 - |z_1 z_2| + |z_1 - z_2|\right)^2} \right|, \quad |z_1| = |z_2| = r, 0 < r < 1.
\]

Now, in general

\[
\left| \sum_{n=1}^{\infty} a_n z^n - \sum_{n=1}^{\infty} a_n z^n \right| \geq \left| f(z_1) - f(z_2) \right| - \left| \sum_{n=1}^{\infty} a_n (z^{n+1} - z_1^{n+1}) \right|,
\]

so that it is sufficient to prove

\[
\left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} \right| > \left| \sum_{n=1}^{\infty} a_n \frac{z_1^{n+1} - z_2^{n+1}}{z_1 - z_2} \right|, \quad |z_1|, |z_2| < \frac{1}{2}.
\]

Considering the maximum modulus-principle, we have only to prove (2) for \( z_1 = 1/2, z_2 = x/2 \) \( (|x| = 1, x \neq 1) \). In (1), putting \( z_1 = 1/2, z_2 = x/2 \)
On the other hand, the right hand side of the inequality (2) for \( n \geq 4 \)

\[
\left| \frac{f\left(\frac{x}{2}\right) - f\left(\frac{1}{2}\right)}{x - \frac{1}{2}} \right| \leq \frac{1}{1 - 2x} \left| \frac{1}{2} \right|^3 = \frac{12}{49}.
\]

therefore we see that all the sections for \( n \geq 4 \) are univalent in \(|z| < 1/2\).

By the analogous method we can prove the following two theorems.

**Theorem 4.**

Let the function \( f(z) \subset (C) \), then every section of the power series

\[
\sum_{n=1}^{\infty} a_n z^n \quad (n = 1, 2, 3, \ldots)
\]

is univalent in the circle \(|z| < \tau\), where \( \tau \) is

\[
\frac{1}{\sqrt{5}} \leq \tau \leq 1/2,
\]

except for the cases \( n = 3, 4 \). And for \( n = 3, 4 \)
Theorem 5.

Let the function \( f(z) \in \mathbb{C} \), then every section of the power series

\[
\sum_{n=1}^{\infty} a_n z^n \quad (n=1,2,3,\ldots)
\]

is univalent in the circle

\[ |z| < \frac{1}{2} \]

for all \( n \geq 7 \), and the number \( 1/2 \) cannot be replaced by any greater one. And the extremal case can be attained by the section of the first two terms of the function

\[ f(z) = \frac{z}{1 - z}. \]

Theorem 6.

Let the function \( f(z) \in \mathbb{C} \), then \( w = f(z) \) maps \( |z| < 1 \) onto a region \( D \) including the circle \( |w| < \frac{1}{3} \), and the number \( 1/3 \) cannot be replaced by any greater one. And the extremal case can be attained by the function

\[ f(z) = \frac{z}{1 - z + z^3}. \]

Proof.

Let \( D \) be the image domain of \( |z| < 1 \) under the schlicht function \( w = f(z) \), and let \( \xi \) be any boundary point of \( D \), then

\[
\frac{\xi f(z)}{\xi - f(z)} = z + \left( a_2 + \frac{1}{\xi} \right) z^2 + \ldots
\]

will also be regular and schlicht in \( |z| < 1 \) since the linear transformation of a schlicht function leads again a schlicht function, so that by Bieberbach's theorem

\[ |a_2 + \frac{1}{\xi}| \leq 2, \]
that is
\[ \left| \frac{1}{\xi} \right| \leq |a_3| + 2. \]

But, by the assumption $|a_3| \leq 1$, hence we see that
\[ |\xi| \geq \frac{1}{|a_3| + 2} \geq \frac{1}{3}. \]

This shows that any point of the $w$-plane which is not taken by $w = f(z)$ in $|z| < 1$ has at least the distance $1/3$ from the origin. This result is sharp, i.e., the constant $1/3$ cannot be replaced by any greater number. That this is so is shown by the function
\[ f(z) = \frac{z}{1 - z + z^3} = z + z^3 + z^4 + z^6 + z^8 + z^{10} + \cdots. \]

which does not take value $w = -1/3$ in $|z| < 1$.

**Bibliography.**