A Method of Electron Trajectory Tracing with its Application to an Electron Optical Investigation of a Fine Focus X-ray Tube.

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In an axially symmetrical electrostatic potential field, the paraxial ray equation of an electron can be solved analytically when the axial potential is expressed in quadratic form in terms of the distance measured along the symmetry axis. On the other hand, the method of dynamical determination of the electron trajectories was employed by Spear.1) Combining these two methods, we attempt to solve the problem of tracing the electron trajectories in modified Ehrenberg-Spear type X-ray tube2)3) whose cathode potential is kept at negative high tension. The results obtained does not show sufficient small focus size, but it may possible to reduce it by adjusting the mutual position of electrodes and the grid potential.

§1. Method of Electron Trajectory Tracing.

The arrangement of electrodes in X-ray tube is shown schematically in Fig. 1, where

A is anode, F is filament, E is glid cover and G is grid (cathode cylinder). The symmetry axis, connecting the center of the filament and of the anode, is taken as z axis with its origin at the filament, and the distance from z-axis is denoted by r. We devide the

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electron optical field in this tube into two regions: the region near the cathode (region (1)) and the region apart from the cathode where the paraxial ray equation is valid (region (2)). Following treatments are possible if we assume the axial potential \( \phi \) as a quadratic form of \( z \):

\[
\phi(z) = az^2 + bz + c. \tag{1}
\]

(1) In region (1) the equations of motion of an electron are

\[
m\ddot{z} = e \frac{\partial V}{\partial z}, \quad m\ddot{r} = e \frac{\partial V}{\partial r}, \tag{2}
\]

where \( e = 4.8 \times 10^{-10} \) e.s.u., \( m = 9.1 \times 10^{-31} \) g and \( V \) is the potential as a function of \( r \) and \( z \). Using Eq. (1) and the well-known relation

\[
V(r,z) = \sum_{n=0}^{\infty} \left( \frac{\phi(2n)}{(n!)^2} \left( \frac{r}{2} \right)^{2n} \right), \tag{3}
\]

\( V \) becomes as follows

\[
V = az^2 + bz + c - (1/2)ar^2. \tag{4}
\]

Therefore Eqs. (2) reduce to

\[
m\ddot{z} = e(2az + b) \tag{5}
\]

and \( m\ddot{r} = -ear \tag{6} \)

If we put \( l^2 = (e/m)a \), the solutions of Eq. (5) are

\[
z = A \exp(\sqrt{2}lt) + B \exp(-\sqrt{2}lt) - \frac{b}{2a} \tag{7}
\]

and \( \dot{z} = A\sqrt{2}l \exp(\sqrt{2}lt) - B\sqrt{2}l \exp(-\sqrt{2}lt) \tag{8} \)

\[
(A = \frac{z_0}{2} + \frac{b}{4a} + \frac{\dot{z}_0}{2l\sqrt{2}}, \quad B = \frac{z_0}{2} + \frac{b}{4a} - \frac{\dot{z}_0}{2l\sqrt{2}}),
\]

where the initial conditions \( z = z_0 \) at \( t = 0 \) and \( z = z_0 \) at \( t = 0 \) are employed. On the other hand, Eq. (6) leads to

\[
r = (r_0/l) \sin lt + r_0 \cos lt \tag{9}
\]

and \( \dot{r} = r_0 \cos lt - r_0 l \sin lt \tag{10} \)

where \( r_0 \) and \( \dot{r}_0 \) are the values of \( r \) and \( \dot{r} \) at \( t = 0 \) respectively. The slope of the
trajectory \(dr/dz\) is given by the relation

\[ dr/dz = -\frac{\dot{r}}{z} \]  

(11)

Thus the electron trajectory tracing in region (1) is performed using Eqs. (8), (9), (10) and (11).

(II) In region (2) the ray equation is expressed as

\[ r'' + \frac{1 + (r')^2}{2V} \frac{\partial V}{\partial z} r' - \frac{1 + (r')^2}{2V} \frac{\partial V}{\partial r} = 0, \]  

(12)

where the prime indicates differentiation with respect to \(z\). If \(r'\ll 1\), and if we neglect \(\frac{1}{2} ar\) in \(V\), (12) becomes

\[ r'' + \frac{2az + b}{2\phi} r' + \frac{a}{2\phi} r = 0. \]  

(13)

By substitution

\[ x = \frac{1}{\sqrt{a}} \log \left\{ \left( az^2 + bz + c \right)^{1/2} + z \sqrt{a} \right\}, \quad \text{for } a > 0 \]  

(14)

\[ x = \frac{-1}{\sqrt{a}} \sin \frac{1}{\sqrt{b^2 - 4ac}} \frac{2az + b}{\sqrt{b^2 - 4ac}}. \quad \text{for } a < 0 \]  

(15)

(13) reduces to

\[ \frac{d^2r}{dx^2} + \frac{2}{a} r = 0. \]  

(16)

With \(a > 0\), if we put \(l^2 = -\frac{2}{a}\), the solutions of Eq. (16) are

\[ r = A \sin lx + B \cos lx \]  

(17)

and \(r' = l(A \cos lx - B \sin lx) \frac{dx}{dz}\).  

(18)

The constants \(A\) and \(B\) are determined from the conditions \(r = r_0\) at \(z = z_0\) (where \(x = x_0\)) and \(r' = r_0'\) at \(z = z_0\), giving

\[ A = r_0 \sin lx_0 + \frac{r_0' \phi_0}{l} \sin lx_0. \]  

(19)

\[ B = r_0 \cos lx_0 - \frac{r_0' \phi_0}{l} \cos lx_0. \]  

(20)

where \(\phi_0\) is the value of \(\phi\) at \(z = z_0\). On the other hand, for
\( q < 0 \), (16) leads to
\[
r = A \exp l x + B \exp (-l x) \tag{21}
\]
and
\[
r' = l x \{ A \exp l x - B \exp (-l x) \} \frac{dx}{dz}. \tag{22}
\]
where \( l^2 = \frac{a}{2} \), and the constants \( A \) and \( B \) are determined from the conditions
\[
r = r_0 \text{ at } z = z_0 \quad \text{and} \quad r' = r_0' \text{ at } z = z_0.
\]

The solutions (17), (18) or (21), (22) are applied only to the fundamental rays, and any electron trajectory which has the given conditions at cathode will be determined by the linear combination of these fundamental rays.

(III) The procedures computing the electron path, employing the above results, are as follows. First, the measured axial potential must be expressed in quadratic form. To do this, we have to determine the coefficients \( a, b, \) and \( c \) in equation (1) such that to fit the measured potential values at three measuring points for each set \( (z_0, z_1, z_2, ...) \), \( (z_3, z_4, z_5, ...) \) etc. The axial potential is measured at points with constant interval (in our case the interval is taken as 1mm, and the measuring points are expressed as \( z_0, z_1, \ldots, z_n \), where \( z_n \) is the position of the target). Second, in region (1), using Eqs. (8), (9), (10) and (11), we get a trajectory of an electron with given initial conditions. In present case region (1) extends from cathode to \( z_b \), and hence we must apply thrice these equations. Last, to obtain the trajectory in region (2), we trace the trajectories of fundamental rays which have any values of \( r \) and \( r' \) at \( z = z_6 \). To do this we solve the paraxial ray equations successively, that is, in the first place, we find the solution in the interval \( z_6 \sim z_8 \), and then the values of \( r \) and \( r' \) at \( z_8 \), serve to determine the solution in \( z_8 \sim z_{10} \). Taking such method successively we can obtain the fundamental rays finally. For the sake of linearity of the paraxial ray equations, we can transform the values of \( r \) and \( r' \) at \( z_8 \) to the values at \( z_8 \) using the linear combination of the fundamental rays. Repeating these procedures, we obtain the trajectory of an electron from \( z_b \) to \( z_N \), thus the complete description of an electron path is performed.


The potential distribution is measured by the usual electric tank method. Mapping of equipotential lines and measurements of axial potentials for various values of \( d \) and \( d_{CG} \) are done. Some of the curves of measured axial potentials versus \( z \) are given in Fig. 2. In these cases the grid is kept at 400 volts negative with respect to the cathode at
the anode potential value of 30 kilo-volts.

As Spear has pointed out, in his “system II” type focusing system in which the cathode is kept negative high tension, a strong field penetrates into the cathode cylinder. This occurs in our tube which resembles to Spear’s “system II”, but this field penetration may be partly removed by increasing the value of $d$. Moreover, in his “system II” the axial potential curve has a remarkable inflection in the region out of the cathode cylinder which tends to defocus the electron beam, but striking inflection does not occur in our case, though slight one is present.

§8. Results of Trajectory Tracing.

Fig. 3. The electron trajectories (for $d=30\text{mm}$, $d_{cg}=0$, grid potential is 400V).

Fig. 4. The electron trajectories (for $d=28\text{mm}$, $d_{cg}=0$, grid potential is 400V).
The electron paths obtained are shown in Fig. 3 for \( d = 30 \text{mm} \), and in Fig. 4 for \( d = 28 \text{mm} \), and the other conditions are same in both cases. The trajectories are traced for the electrons whose initial velocities are \( 1.2 \times 10^7 \text{cm/sec} \) (slow electron), \( 3.6 \times 10^7 \text{cm/sec} \) (most probable velocity) and \( 6 \times 10^7 \text{cm/sec} \) (fast electron), emitted from \( r_0 = 0.00 \text{mm} \) and \( r_0 = 0.05 \text{mm} \), where \( r_0 \) is the value of \( r \) at the cathode. The angles of emission are chosen as \( \theta = 6^\circ, 18^\circ \), and \( 42^\circ \). The group of three numbers marked above the trajectory denotes the following conditions: the first number represents the value of \( r_0 \) in mm, the second gives the initial velocity in units of \( 10^7 \text{ cm/sec} \) and the third indicates the angle of emission.


Now we shall discuss the following two problems: first, the accuracy of this method, second, the results obtained.

(i) In our method the axial potential is approximated to a group of the segments which have the quadratic forms of \( z \), but the real potential should deviate from such form. This error is not so serious because the axial potential curve resembles to a quadratic form as a whole. Rather an important matter is that an approximated axial potential curve has discontinuous differential coefficients with respect to \( z \) at the boundaries of the segments. But this occur in other approximational method of trajectory tracing. It may be said that our method is not inferior to the other methods in this point of view, because the effect of discontinuous force in \( r \) direction at the boundary of segments and the action of continuously changing force in \( r \) and \( z \) directions in one segment are automatically included.

Next, in paraxial ray equations the terms \( r'^2 \) and \( \frac{1}{2} ar^2 \) are neglected. However, the maximum value of \( r'^2 \) is less than \( 10^{-4} \), and the maximum ratio of \( \frac{1}{2} ar^2 \) to \( \phi \) is about \( 10^{-3} \), so that this is less than the experimental error of \( \phi \). Indeed the comparison of a trajectory obtained by the dynamical method with that obtained by the paraxial ray equation in region (2) shows about 0.1% difference. The dominant error in our tracing method is that introduced by the unaccuracy in the measurement of \( \phi \), especially in the region near cathode. In this region the experimental error of \( \phi \) is estimated as 3%, and it may affect considerably the shapes of the trajectories. It should be required to increase this accuracy. The other errors introduced by neglecting the asymmetric shape of the filament and the relativistic correction should not be so large.
(ii) The electron trajectories shown in Fig. 3 and 4 resemble to rather those in Spear's "system I" type focusing system than those in his "system II", though in detail they are different, and the focal sizes of electron beams in Fig. 3 and 4 are not so small as Spear's. It may be predicted, therefore, that the electron optical system shown in Fig. 1 is suitable for use in a fine focus X-ray tube. To ensure this fact, we should investigate the effects of the mutual position of the electrodes and the grid potential on the focal size. The focusing system with \(d=30\text{mm}\) gives smaller focal size than with \(d=28\text{mm}\). This difference perhaps relates to the difference in the shapes of the axial potential curve in the region \(z_6 \sim z_{10}\). If we reduce the value of \(d\), the trajectories give increased tendency shown in Fig. 4, so that the focal size becomes larger. But only from present investigations it is not reasonable to conclude that the focusing system with \(d=30\text{mm}\) is the optimum one. The paper will be published in which more detailed studies on these problems will be described.

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References

3) N. Shibata: Sci. Rep. of the Faculty of Arts and Literature, Nagasaki University, No.4 (1955) 25.