Research on the Tertiary Scattering of the Sun’s Ray in the Earth’s Atmosphere

Takao Sato
(Nagasaki University)

Abstract: Research on the tertiary scattering of the Sun's ray has not yet been produced. In this paper, the author will instruct its fundamental principle and method of computation.

Chapter 1. Fundamental Principle of the Tertiary Scattering.

In this chapter, see fig. 1.
Take three points \( F, T \) and \( E \) in the atmosphere. Define a system of rectangular coordinates \( X_1, Y_1, Z_1 \), with its center at \( F \), with the \( X_1 \) axis drawn towards the sun. Let \( \tau_i, \delta_i, \kappa_i \) be the coordinates of \( T \) of this system.

Now, if \( i \) be the direct insolation reaching \( F \), we can resolve it into two plane polarized rays, the one is

\[
E(Z_1) = \sqrt{\frac{i}{2}} \exp(i \tau + i \delta)
\]

which travels to \( X_1 \) and oscillates in \( Z_1 \) direction, the other is

\[
E(Y_1) = \sqrt{\frac{i}{2}} \exp(i \tau + i \delta)
\]

travelling to \( X_1 \) and oscillating in \( Y_1 \) direction.

Here, \( FT = R_1 \) and \( \varepsilon \) be the dielectric constant of air particles. Then the equation of wave generated at \( T \) when the former encounters one air particle at \( F \) is expressed by

\[
E_{is} = -\sqrt{\frac{i}{2}} \frac{\pi T}{R_1 \lambda^2} \varepsilon \left( \frac{1}{\varepsilon} \right) \frac{\tau_i \kappa_i}{R_1^2} \exp(i \tau - i \delta)
\]

\[
E_{iv} = \sqrt{\frac{i}{2}} \frac{\pi T}{R_1 \lambda^2} \varepsilon \left( \frac{1}{\varepsilon} \right) \frac{\delta_i \kappa_i}{R_1^2} \exp(i \tau - i \delta)
\]

\[
E_{it} = -\sqrt{\frac{i}{2}} \frac{\pi T}{R_1 \lambda^2} \varepsilon \left( \frac{1}{\varepsilon} \right) \frac{\tau_i^2 + \delta_i^2}{R_1^2} \exp(i \tau - i \delta)
\]

in which \( \lambda \) and \( T \) are the wave length and the volume of particle. This wave will proceed into the direction \( FT \) and have a direction of oscillation normal to it, and the direction cosines of the latter are as follows:

\[
\sin \omega_i = \frac{\sqrt{\tau_i^2 + \delta_i^2}}{R_1},
\]

\[
\frac{1}{\sin \omega_i} = \frac{\tau_i}{R_1}, \quad \frac{1}{\sin \omega_i} = \frac{\delta_i}{R_1}, \quad \frac{1}{\sin \omega_i} = \frac{\tau_i^2 + \delta_i^2}{R_1^2};
\]

i.e. \( \omega_i \) is the angle between \( FT \) and \( Z_1 \) axis. The intensity can be obtained by squaring the amplitude, which is

\[
E_i = \sqrt{\frac{i}{2}} \frac{\pi T}{R_1 \lambda^2} \varepsilon \left( \frac{1}{\varepsilon} \right) \sin \omega_i
\]

In the same way, the plane polarized light generated at \( T \) by \( E(Y_1) \) is
\[
E'_{1z} = \frac{i}{2} \frac{\pi T}{R_1 \lambda} \varepsilon (A \frac{1}{\varepsilon}) \frac{\gamma_1 \delta_1}{R_1^2} \exp(int - im R_1),
\]

\[
E'_{1y} = -\frac{i}{2} \frac{\pi T}{R_1 \lambda} \varepsilon (A \frac{1}{\varepsilon}) \frac{\gamma_1^2 + \varepsilon_1^2}{R_1^2} \exp(int - im R_1),
\]

\[
E'_{1x} = \frac{i}{2} \frac{\pi T}{R_1 \lambda} \varepsilon (A \frac{1}{\varepsilon}) \frac{\delta_1 \varepsilon_1}{R_1^2} \exp(int - im R_1).
\]

This will also proceed into \( FT \) direction and oscillate normal to it, the amplitude being

\[
E'_1 = \frac{i}{2} \frac{\pi T}{R_1 \lambda} \varepsilon (A \frac{1}{\varepsilon}) \sin \omega t'.
\]

in which \( \omega ' \) is the angle between \( FT \) and \( Y_1 \) axis. In the following discussion, let the amplitudes \( E_1 \) and \( E'_1 \) also be the notations of their corresponding polarizations.

Now take the origin at \( T, X_2 \) axis in \( TF \) direction, \( Z_2 \) axis in the direction of oscillation of \( E_1 \), \( Y_2 \) axis normal to \( X_2Z_2 \). In the same way, define \( X'_2 \), \( Y'_2 \), \( Z'_2 \) with respect to \( E'_1 \), in which \( X'_2 \) being identical with \( X_2 \).

Now, take a point \( E \), and let \( TE = R_2 \) and the angular distances of \( TE \) from \( Z_2 \) and \( Z'_2 \) axes be \( \omega_2 \) and \( \omega'_2 \). Then, \( E_1 \) at \( T \) will also generate one plane polarized light at \( E \). Letting the coordinates of \( E \) referred to \( X_2Y_2Z_2 \) system be \( \gamma_2 \delta_2 \varepsilon_2 \), then this new plane polarized light will be

\[
E_{z2} = E'_1 \frac{\pi T}{R_2 \lambda} \varepsilon (A \frac{1}{\varepsilon}) \frac{\gamma_2 \delta_2 \varepsilon_2}{R_2^2} \exp(int - im R_2),
\]

\[
E_{y2} = E'_1 \frac{\pi T}{R_2 \lambda} \varepsilon (A \frac{1}{\varepsilon}) \frac{\delta_2 \varepsilon_2}{R_2^2} \exp(int - im R_2),
\]

\[
E_{x2} = -E'_1 \frac{\pi T}{R_2 \lambda} \varepsilon (A \frac{1}{\varepsilon}) \frac{\gamma_2^2 + \delta_2^2}{R_2^2} \exp(int - im R_2),
\]

whose amplitude being

\[
E_2 = E'_1 \frac{\pi T}{R_2 \lambda} \varepsilon (A \frac{1}{\varepsilon}) \sin \omega_2.
\]

In the same way, the plane polarized light generated by \( E'_1 \) at \( E \) point can be expressed by substituting \( E_2 E'_1 \gamma_2 \delta_2 \varepsilon_2 \omega_2' \) for \( E_2 E'_1 \gamma_2 \delta_2 \varepsilon_2 \omega_2 \) in the above expressions with respect to \( X_2Y_2Z_2 ' \) system. Here, both directions of oscillation of \( E_2 \) and \( E'_2 \) are naturally normal to \( TE \). Now, taking the origin at \( E, X_3 \) axis in \( ET \) direction, \( Z_3 \) in the direction of oscillation of \( E_3 \), \( Y_3 \) normal to them, \( X'_3 \) in \( ET \) direction, \( Z'_3 \) in the
direction of oscillation of $E_1', Y_3'$ normal to them, and letting the coordinates of $O'$ with respect to these systems be $\gamma_3 \delta \kappa_3, \gamma_3' \delta \kappa_3', EO'=R_3$ and the angular distances of $EO'$ from $Z_3$ and $Z_3'$ axes be $\omega_3$ and $\omega_3'$, then the plane polarized light generated at $O'$ by $E_2$ is expressed by referring $X_3'Y_3'Z_3'$

$$E_2 = E_2 \frac{\pi T}{R_3^2} \epsilon \left( \frac{1}{\epsilon} \right) \frac{r_{23}^2 \kappa_3}{R_3^3} \exp(i \tau - i m R_3),$$

$$E_2 = E_2 \frac{\pi T}{R_3^2} \epsilon \left( \frac{1}{\epsilon} \right) \frac{\delta_{23} \kappa_3}{R_3^3} \exp(i \tau - i m R_3),$$

$$E_2 = -E_2 \frac{\pi T}{R_3^2} \epsilon \left( \frac{1}{\epsilon} \right) \frac{r_{23}^2 + \delta_{23}^2}{R_3^3} \exp(i \tau - i m R_3),$$

whose amplitude being

$$E_2 = E_2 \frac{\pi T}{R_3^2} \epsilon \left( \frac{1}{\epsilon} \right) \sin \omega_3.$$

In the same way, the plane polarized light generated by $E_2'$ at $O'$ point can be expressed by substituting $E_1' \gamma_3 \delta \kappa_3 \omega_3'$ for $E_1 \gamma_3 \delta \kappa_3 \omega_3$ in the above expressions with respect to $X_3'Y_3'Z_3'$ system.

Eventually we can get

$$E_2 = \sqrt{\frac{i}{2}} \left( \frac{\pi T}{\lambda^2} \epsilon \left( \frac{1}{\epsilon} \right) \right) \frac{\sin \omega_1}{R_1} \frac{\sin \omega_2}{R_2} \frac{\sin \omega_3}{R_3},$$

$$E_2' = \sqrt{\frac{i}{2}} \left( \frac{\pi T}{\lambda^2} \epsilon \left( \frac{1}{\epsilon} \right) \right) \frac{\sin \omega_1'}{R_1} \frac{\sin \omega_2'}{R_2} \frac{\sin \omega_3'}{R_3}.$$

Now, let the number of particles in unit volume at $F, T$ and $E$ be $T_F, T_T$ and $T_E$ respectively and put

$$k_{\lambda n} = \frac{8 \pi^3 T^3}{3 \lambda^4} \Gamma_n \epsilon^2 \left( \frac{1}{\epsilon} \right)^3.$$

(\( n = F, T, E \)).

As the energy is the square of the amplitude, so the intensity of tertiary scattering generated at $O'$ by unit volume of $F, T$ and $E$ will be

$$\frac{i}{2} \left( \frac{\pi T}{\lambda^2} \epsilon \left( \frac{1}{\epsilon} \right) \right)^6 \frac{1}{K_1^3} \frac{1}{K_2^3} \frac{1}{R_3^3} \left( \sin^2 \omega_1 \sin^2 \omega_2 \sin^2 \omega_3 + \sin^2 \omega_1' \sin^2 \omega_2' \sin^2 \omega_3' \right) k_{\lambda F} k_{\lambda T} k_{\lambda E}.$$
the absorption terms.

Chapter 2. The practical method of calculation.

Let \(O\) be the earth's center and \(O'\) a point on its surface. Take a coordinate system \(XYZ\) with its origin at \(O\), \(Z\) axis being directed towards \(OO'\), \(X\) axis normal to \(Z\) axis and towards the Sun's side on the plane containing \(OO'\) and its center, \(Y\) axis normal to \(X\) and \(Z\).

Take a point \(E\) in the atmosphere seen from \(O'\) and let the coordinates referred to \(XYZ\) system be

\[X = e \sin \gamma \cos A, \quad Y = e \sin \gamma \sin A, \quad Z = e \cos \gamma,\]

in which

\[0 \leq \gamma \leq \frac{\pi}{2}, \quad 0 \leq A \leq 2\pi.\]

Let \(X'Y'Z'\) system be the parallel translation of \(XYZ\) system by the transformation of the origin \(O\) to \(O'\), and put

\[X' = R \cos \theta \cos A, \quad Y' = R \cos \theta \sin A, \quad Z' = R \sin \theta.\]

Letting the coordinates of \(E\) referred to this new system be \(X'Y'Z'\), so we get

\[X = X', \quad Y = Y', \quad Z = Z' + a_0,\]

here \(a_0\) being the earth's radius.

The definition of \(x_2y_2z_2\) axes in relation to \(XYZ\) axes was already given in the previous paper (1). Now, let the coordinates of \(T\) referred to \(x_2y_2z_2\) system be

\[x_{2T} = OT \sin \theta_4 \cos A_1, \quad y_{2T} = OT \sin \theta_4 \sin A_1, \quad z_{2T} = OT \cos \theta_4,\]

(see fig. 2)

Let \(x'_2y'_2z'_2\) system be the parallel transformation of \(x_2y_2z_2\) system by the transformation of the origin \(O\) to \(E\), and put \(\angle EOT = \theta_2, \angle OET = \theta_1\), then
\[
\sin \theta_2 = \frac{ET \sin \theta_1}{OT},
\]
and the coordinates of \(T\) referred to \(x_2'y_2'z_2'\) will be
\[
x'_2 = ET \sin \theta_1 \cos A_1, \quad y'_2 = ET \sin \theta_1 \sin A_1, \quad z'_2 = -ET \cos \theta_1.
\]
Giving \(ET, \theta_1A_1\), we can get \(OT\) in the same way as in the paper (2), in which the method of obtaining \(ET\) is also given, so \(\theta_2\) can be found, and eventually \(x_2'y_2'z_2\).

Moreover, let the polar coordinates of \(T\) referred to \(XYZ\) system be
\[
X_T = OT \sin \gamma' \cos A', \quad Y_T = OT \sin \gamma' \sin A', \quad Z_T = OT \cos \gamma'.
\]

The combination of the relation between \(XYZ\) and \(x_2'y_2'z_2\) systems

<table>
<thead>
<tr>
<th>(x_2)</th>
<th>(y_2)</th>
<th>(z_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X)</td>
<td>(\cos \gamma \cos A)</td>
<td>(-\sin A)</td>
</tr>
<tr>
<td>(Y)</td>
<td>(\cos \gamma \sin A)</td>
<td>(\cos A)</td>
</tr>
<tr>
<td>(Z)</td>
<td>(-\sin \gamma)</td>
<td>(O)</td>
</tr>
</tbody>
</table>

and the polar coordinates of \(x_2'y_2'z_2\) system will generate the following formulae
\[
\sin \gamma' \cos A' = \cos \gamma \cos A \sin \theta_2 \cos A_1 - \sin A \sin \theta_2 \sin A_1 + \sin \gamma \cos A \cos \theta_2,
\]
\[
\sin \gamma' \sin A' = \cos \gamma \sin A \sin \theta_2 \cos A_1 + \cos A \sin \theta_2 \sin A_1 + \sin \gamma \sin A \cos \theta_2,
\]
\[
\cos \gamma' = -\sin \gamma \sin \theta_2 \cos A_1 + \cos \gamma \cos \theta_2.
\]

From this we can find \(\gamma'\) and \(A'\) if the right sides are known.

Define a new system of coordinates \(x_3'y_3z_3\) by the next relation, here \(z_3\) axis is directed to \(OT\). (see fig. 3)
Now, let the coordinates of \( F \) referred to \( x_3y_3z_3 \) system be

\[
\begin{align*}
x_3'F &= OF \sin \theta_4 \cos A_2, \\
y_3'F &= OF \sin \theta_4 \sin A_2, \\
z_3'F &= OF \cos \theta_4.
\end{align*}
\]

(see fig. 4.)

Further, let \( x_3'y_3'z_3' \) be the parallel translation of \( x_3y_3z_3 \) system by the transformation of \( O \) to \( T \) (see fig. 4), and the coordinates of \( F \) referred to this new system be

\[
\begin{align*}
x_{3p}' &= TF \sin \theta_4 \cos A_2, \\
y_{3p}' &= TF \sin \theta_4 \sin A_2, \\
z_{3p}' &= -TF \cos \theta_4.
\end{align*}
\]

So, if \( \theta_4, A_2 \) are given and \( TF \) are calculated, the position of \( F \) are determined, and in the same way as in \( T \), its height \( OF \) can be obtained. From this and

\[
\sin \theta_4 = \frac{FT}{OF} \sin \theta_4 \theta,
\]

we can get \( \theta_4 \) and so \( x_{3p}', y_{3p}', z_{3p}' \) can be determined.

However the coordinates of \( T \) referred to \( x_3'y_3'z_3' \) system are

\[
\begin{align*}
x_{3p}' &= y_{3p}' = 0; \\
z_{3p}' &= OT.
\end{align*}
\]

By giving these values we can get the coordinates of \( F, T \) referred to \( XYZ \) system
can be determined.

Therefore, the coordinates of $T$ referred to $X_1Y_1Z_1$ system can be obtained as follows:

$$r_1 = (X'_T - X'_F) \cos h + (Z'_T - Z'_F) \sin h,$$
$$\delta_1 = Y'_T - Y'_F,$$
$$\kappa_1 = (Z'_T - Z'_F) \cos h - (X'_T - X'_F) \sin h.$$ 

The direction cosine of $FT$ referred to $X_1, Y_1, Z_1$, axes are

$$\cos \omega''_1 = \frac{r_1}{FT}, \quad \cos \omega'_1 = \frac{\delta_1}{FT}, \quad \cos \omega_1 = \frac{\kappa_1}{FT},$$

in which $\omega''_1$ is the angle between $FT$ and $X_1$ axis.

In the expressions of $r_1$, $\delta_1$, $\kappa_1$, $Z'_T - Z'_F = Z_T - Z_F$, $X'_T - X'_F = X_T - X_F$,

$$Y'_T - Y'_F = Y_T - Y_F.$$ 

They are linear functions of $x_{3F} - x_{3F}$, $y_{3F} - y_{3F}$, $z_{3F} - z_{3F}$, in which

$$x_{3F} - x_{3F} = x_{3F} - x_{3F} = -x_{3F} = -TF \sin \theta_3 \cos \alpha,$$
$$y_{3F} - y_{3F} = y_{3F} - y_{3F} = -y_{3F} = -TF \sin \theta_3 \sin \alpha,$$
$$z_{3F} - z_{3F} = OT - OF \cos \theta_3 = TF \cos \alpha.$$ 

by

$$X_F - X_F = \cos \gamma' \cos A'(x_{3F} - x_{3F}) - \sin A'(y_{3F} - y_{3F}) + \sin \gamma' \cos A'(z_{3F} - z_{3F})$$
$$Y_F - Y_F = \cos \gamma' \sin A'(x_{3F} - x_{3F}) + \cos A'(y_{3F} - y_{3F}) + \sin \gamma' \sin A'(z_{3F} - z_{3F})$$
$$Z_F - z_F = -\sin \gamma' (x_{3F} - x_{3F}) + \cos \gamma' (z_{3F} - z_{3F}).$$

Therefore, $\cos \omega_1$, $\cos \omega'', \cos \omega''', \theta_3$, can be evaluated by sine and cosine of $\gamma' A' \theta_3 A_3$.

The coordinates of $E$ referred to $X_1Y_1Z_1$ system are

$$X_{1E} = (X'_E - X'_F) \cos h + (Z'_E - Z'_F) \sin h,$$
$$Y_{1E} = Y'_E - Y'_F,$$
$$Z_{1E} = (Z'_E - Z'_F) \cos h - (X'_E - X'_F) \sin h.$$ 

Then

$$X_{1E} - r_1 = (X'_E - X'_T) \cos h + (Z'_E - Z'_T) \sin h = (X_E - X_T) \cos h + (Z_E - Z_T) \sin h,$$
$$Y_{1E} - \delta_1 = Y'_E - Y'_T = Y_E - Y_T,$$
$$Z_{1E} - \kappa_1 = (Z'_E - Z'_T) \cos h - (X'_E - X'_T) \sin h = (Z_E - Z_T) \cos h - (X_E - X_T) \sin h,$$

in which
\[ X_E - X_T = l_1(x'_2E - x'_2T) + l_2(y'_2E - y'_2T) + l_3(z'_2E - z'_2T), \]
\[ Y_E - Y_T = m_1(x'_2E - x'_2T) + m_2(y'_2E - y'_2T) + m_3(z'_2E - z'_2T), \]
\[ Z_E - Z_T = n_1(x'_2E - x'_2T) + n_2(y'_2E - y'_2T) + n_3(z'_2E - z'_2T). \]

Here, \( x'_2E, y'_2E, z'_2E \) being zero.

Eventually
\[
\frac{X_E - X_T}{ET} = -\cos \gamma \cos A \sin \theta_1 \cos A_1 + \sin A \sin \theta_1 \sin A_1 + \sin \gamma \cos A \cos \theta_1,
\]
\[
\frac{Y_E - Y_T}{ET} = -\cos \gamma \sin A \sin \theta_1 \cos A_1 - \cos A \sin \theta_1 \sin A_1 + \sin \gamma \sin A \cos \theta_1,
\]
\[
\frac{Z_E - Z_T}{ET} = \sin \gamma \sin \theta_1 \cos A_1 + \cos \gamma \cos \theta_1,
\]

From this, the direction cosines of $TE$ line referred to $X_1Y_1Z_1$ system can be evaluated from
\[
\frac{X_{1E} - r_1}{TE} = \frac{Y_{1E} - \delta_1}{TE}, \quad \frac{Z_{1E} - \kappa_1}{TE} \quad \text{by sine and cosine of } \gamma, A, \theta_1, A_1.
\]

For abridgement we will write them by $l, m, n$.

Further, let the direction cosines of $TZ_2$ and $TZ'_2$ referred to $X_1Y_1Z_1$ system be $l_2m_2n_2$ and $l'_2m'_2n'_2$, so we have
\[
\frac{\tau_1\kappa_1}{K_1^2} = \cos \omega_1 \cos \omega_1'' = l_2 \sin \omega_1 = L_1,
\]
\[
\frac{\delta_1\kappa_1}{K_1^2} = \cos \omega_1' \cos \omega_1 = n_2 \sin \omega_1 = M_1,
\]
\[- \frac{\tau_1^2 + \delta_1^2}{K_1^2} = -\left( \cos^2 \omega_1' + \cos^2 \omega_1'' \right) = -1 + \cos^2 \omega_1 = \kappa_2 \sin \omega_1 = N_1,
\]
\[
\cos \omega_2 = \mu_2 + m m_2 + n n_2.
\]

And
\[
\frac{\tau_1\delta_1}{K_1^2} = \cos \omega_1'' \cos \omega_1' = l_2' \sin \omega_1' = L_2,
\]
\[
\frac{-\tau_1^2 + \delta_1^2}{K_1^2} = -\left( \cos^2 \omega_1 + \cos^2 \omega_1'' \right) - 1 + \cos^2 \omega_1 = m_2' \sin \omega_1' = M_2,
\]
\[
\frac{\delta_1\kappa_1}{K_1^2} = \cos \omega_1' \cos \omega_1 = n_2' \sin \omega_1' = N_2.
\]
\[
\cos \omega_2' = H_2' + m_2'n_2' + mn_2'.
\]

From these relations
\[
\cos \omega_1 \sin \omega_1 = lL_1 + mM_1 + nN_1,
\]
\[
\sin^2 \omega_1 \sin^2 \omega_2 = \sin^2 \omega_1 - (lL_1 + mM_1 + nN_1)^2.
\]

In the same way
\[
\sin^2 \omega_1' \sin^2 \omega_2' = \sin^2 \omega_1' - (l'_{2} + m'M_2 + nN_2)^2.
\]

The direction cosines of \( X_2 \) axis referred to \( X_1Y_1Z_1 \) system are
\[
-\frac{\gamma_1}{R_1}, \quad -\frac{\delta_1}{R_1}, \quad -\frac{\kappa_1}{R_1},
\]

and that of \( Z_2 \) axis \( l_2m_2n_2 \), then that of \( Y_2 \) can be determined. Let them be \( \lambda, \mu, \nu \)

then we get

\[
\begin{array}{ccc}
X_2 & Y_2 & Z_2 \\
X_1 & -\frac{\gamma_1}{R_1} & \lambda & l_2 \\
Y_1 & -\frac{\delta_1}{R_1} & \mu & m_2 \\
Z_1 & -\frac{\kappa_1}{R_1} & \nu & n_2
\end{array}
\]

Similarly

\[
\begin{array}{ccc}
X'_2 & Y'_2 & Z'_2 \\
X_1 & -\frac{\gamma'_1}{R_1} & \lambda' & l'_2 \\
Y_1 & -\frac{\delta_1}{R_1} & \mu' & m'_2 \\
Z_1 & -\frac{\kappa_1}{R_1} & \nu' & n'_2
\end{array}
\]

From these relations we have
\[
X_1 = \gamma_1 - \frac{\gamma_1}{R_1} X_2 + \lambda Y_2 + l_2 Z_2,
\]
\[
Y_1 = \delta_1 - \frac{\delta_1}{R_1} X_2 + \mu Y_2 + m_2 Z_2,
\]
\[ Z_1 = \kappa_1 - \frac{\kappa_1}{R_1} X_2 + \nu Y_2 + r_2 Z_2, \]
\[ X_1 = \tau_1 - \frac{\tau_1}{R_1} X'_2 + \lambda' Y'_2 + l'_2 Z'_2, \]
\[ Y_1 = \delta_1 - \frac{\delta_1}{R_1} X'_2 + \mu' Y'_2 + m'_2 Z'_2, \]
\[ Z_1 = \kappa_1 - \frac{\kappa_1}{R_1} X'_2 + \nu' Y'_2 + n'_2 Z'_2. \]

Now, the direction cosines of \( Z_3 \) axis referred to \( X_2Y_2Z_2 \) system are

\[ \frac{\gamma_3 \kappa_3}{R_2^2 \sin \omega_2}, \quad \frac{\delta_3 \kappa_3}{R_2^2 \sin \omega_2}, \quad \frac{\gamma_3^2 + \delta_3^2}{R_2^2 \sin \omega_2}. \]

Here \( \tau_2, \delta_2, \kappa_2 \) are the coordinates of \( E \) referred to \( X_2Y_2Z_2 \) system, and can be evaluated by

\[ \tau_2 = -(X_{1B} - \tau_1) \frac{\tau_1}{R_1} - (Y_{1B} - \delta_1) \frac{\delta_1}{R_1} - (Z_{1B} - \kappa_1) \frac{\kappa_1}{R_1}, \]
\[ \delta_2 = -(X_{1B} - \tau_1) \lambda + (Y_{1B} - \delta_1) \mu + (Z_{1B} - \kappa_1) \nu, \]
\[ x_2 = -(X_{1B} - \tau_1) l_2 + (Y_{1B} - \delta_1) m_2 + (Z_{1B} - \kappa_1) n_2. \]

Hence, the above direction cosines can be also evaluated. When we denote them by \( l_m n_m s_m \), the values referred to \( X_1Y_1Z_1 \) system become

\[ - \frac{\tau_1}{R_1} l_3 + \lambda m_3 + l_3 s_3, \]
\[ - \frac{\delta_1}{R_1} l_3 + \mu n_3 + n_3 s_3, \]
\[ - \frac{\kappa_1}{R_1} l_3 + \nu n_3 + n_3 s_3. \]

Let it be \( \xi, \eta, \zeta \).

Next, the direction cosines of \( Z'_3 \) axis referred to \( X'_2Y'_2Z'_2 \) system are

\[ \frac{\gamma'_3 \kappa'_3}{R_2^2 \sin \omega'_2}, \quad \frac{\delta'_3 \kappa'_3}{R_2^2 \sin \omega'_2}, \quad \frac{\gamma'_3^2 + \delta'_3^2}{R_2^2 \sin \omega'_2}. \]

Here \( \gamma'_2, \delta'_2, \kappa'_2 \) are the coordinates of \( E \) referred to \( X'_2Y'_2Z'_2 \), and can be evaluated by

\[ \gamma'_2 = -(X_{1B} - \tau_1) \frac{\tau_1}{R_1} - (Y_{1B} - \delta_1) \frac{\delta_1}{R_1} - (Z_{1B} - \kappa_1) \frac{\kappa_1}{R_1}, \]

\[ -23 - \]
\[ \delta' = (X_{1E} - \tau) \nu' + (Y_{1E} - \delta) \mu' + (Z_{1E} - \kappa') \lambda', \]
\[ \kappa' = (X_{1E} - \tau) \mu' + (Y_{1E} - \delta) \lambda' + (Z_{1E} - \kappa) \nu'. \]

Hence, the above direction cosines can be also evaluated. When we denote them by \( l'_3, m'_3, n'_3 \), the values referred to \( X_1 Y_1 Z_1 \) system become
\[ -\frac{\tau}{R_1} l'_3 + m'_3 + l''_2 n'_3 \]
\[ -\frac{\delta}{R_1} l'_3 + \mu'_3 + m'_2 n'_3 \]
\[ -\frac{\kappa}{R_1} l'_3 + \nu'_3 + n'_2 n'_3 \]

Let it be \( \xi', \eta', \zeta' \).

Besides, the coordinates of \( O' \) referred to \( X_1 Y_1 Z_1 \) system are:
\[ X_1 = -X_F \cos h - Z_F \sin h, \]
\[ Y_0 = -Y_F, \]
\[ Z_0 = -Z_F \cos h + X_F \sin h. \]

So, the direction cosines of \( EO' \) referred to \( X_1 Y_1 Z_1 \) system can be evaluated from
\[ \frac{X_{10}'}{R_1}, \quad \frac{Y_{10}'}{R_3}, \quad \frac{Z_{10}'}{R_3}, \]
these being denoted by \( \lambda_1, \mu_1, \nu_1 \).

Eventually \( \cos \omega_3 \) and \( \cos \omega'_3 \) can be also evaluated from
\[ \cos \omega_3 = \xi_1 + \eta_1 + \zeta_1 \nu_1, \]
\[ \cos \omega'_3 = \xi_1 + \eta'_1 + \zeta'_1 \nu_1. \]

Hence we can evaluate
\[ \sin^2 \omega_1 \sin^2 \omega_2 \sin^2 \omega_3 + \sin^2 \omega_1 \sin^2 \omega'_1 \sin^2 \omega'_2 \sin^2 \omega'_3. \]

The discussions for the absorption terms is the same as in the paper (2).

This research is being continued by the aid of the Scientific Research Expenditure of the Ministry of Education.

The author's Literature


Journal of the Meteorological Society of Japan,

(2) Studies on the Scattering of the Sun's Light by the Earth's Atmosphere.

Science Reports of the Tohoku University