A fractal model for characterizing fluid flow in fractured rock masses based on randomly distributed rock fracture networks

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Abstract:

A fractal model that represents the geometric characteristics of rock fracture networks is proposed to link the fractal characteristics with the equivalent permeability of the fracture networks. The fracture networks are generated using the Monte Carlo method and have a power law size distribution. The fractal dimension $D_T$ is utilized to represent the tortuosity of the fluid flow, and another fractal dimension $D_f$ is utilized to represent the geometric distribution of fractures in the networks. The results indicate that the equivalent permeability of a fracture network can be significantly influenced by the tortuosity of the fluid flow, the aperture of the fractures and a random number used to generate the fractal length distribution of the fractures in the network. The correlation of fracture number and fracture length agrees well with the results of previous studies, and the calculated fractal dimensions $D_f$ are consistent with their theoretical values, which confirms the reliability of the proposed fractal length distribution and the stochastically generated fracture network models. The optimal hydraulic path can be identified in the longer fractures along the fluid flow direction. Using the proposed fractal model, a mathematical expression between the equivalent permeability $K$ and the fractal dimension $D_f$ is proposed for models with large values of $D_f$. The differences in the calculated flow volumes between the models that consider and those that do not consider the influence of fluid flow tortuosity are as high as 17.64% – 19.51%, which emphasizes that the effects of tortuosity should not be neglected and should be included in the fractal model to accurately estimate the hydraulic behavior of fracture networks.
Keywords: Fracture network; Permeability; Fractal dimension; Geometric distribution; Tortuosity
1. Introduction

Permeability is a crucial hydro-mechanical property of rock masses and is important in many areas of geosciences and geoengineering, including dam foundations and petroleum reservoirs. The permeability of a rock mass is mainly governed by rock fractures that separate intact rock blocks with negligible matrix permeability (e.g., granite and basalt) [1-2]. A tremendous amount of effort has been exerted to understand the behavior of fluid flow in rock masses in recent decades [3-7]. However, accurately estimating the permeability of rock masses is still challenging because of the complexities of fracture distributions at the macro-structural (i.e., geometry of the fracture network) and micro-structural (i.e., geometry of the void spaces within single fractures) levels [8-11]. One key difficulty is that rock fractures typically have rough surfaces between which fluid flows non-uniformly. A particle will travel a longer distance along a tortuous path through a rough-walled fracture than through a parallel-walled fracture. Another difficulty is mathematically describing the geometric distributions of rock fractures in fracture networks, which usually contain several sets of fractures with different orientations, lengths and apertures. Fortunately, the distribution of fractures in fracture networks have been found to exhibit fractal characteristics [12-16], which provides a possible approach for describing the geometric characteristics of fracture networks while considering both the macro-scale and micro-scale properties of the fractures.

A few predictive fractal models have been developed to calculate the permeability of stochastic rock fracture networks. The purpose of their models and the outcome of their studies are summarized in Table 1.

Based on the fractal models proposed for the porous media [10] and regular tree networks [21, 27] to calculate the permeability of rock masses, the present study focused on extending this fractal model to the
fractured media consisted of randomly generated stochastic discrete fracture networks (DFNs) using the Monte Carlo method. A probability density function was derived to depict the trace length $l$ of each rock fracture between a minimum and maximum trace length, and the apertures of the fractures were correlated with their trace lengths. Flow simulations of models with various fractal dimensions were conducted, and the relations between the fractal dimensions and the equivalent permeability were estimated.

2. Fractal characteristics of rock fractures

Mandelbrot (1982) [28] verified that the cumulative size distribution of islands on the surface of the earth followed the power law

$$N'(A' > a') \sim a'^{-D/2}$$

(1)

where $N'$ is the total number of islands with an area $A'$ greater than a constant $a'$ and $D$ is the fractal dimension that represents the size distribution of the islands.

Based on this theory, Majumdar and Bhushan [29] developed an equivalent equation to describe the distribution of islands by regarding $a'_{\text{max}}$ as the largest island:

$$N'(A' \geq a') = \left(\frac{a'_{\text{max}}}{a'}\right)^{D/2}$$

(2)

Eq. (2) shows that there is only one largest island on the earth, which is true in the physical world. Xu et al. [21, 27] used Eq. (2) to describe the geometric distribution of pores in porous media that are embedded with randomly distributed 2-D fractal-like tree networks, where $a'_{\text{max}} = g\lambda^2_{\text{max}}$, $a' = g\lambda^2$, $\lambda$ is the diameter of a pore, and $g$ is a geometric factor. The distribution of fractures in 2-D rock masses is considered to be analogous to that of islands on the surface of the earth and that of pores in porous media, which yields
\[ N(L \geq l) = \left( \frac{l_{\text{max}}}{l} \right)^{D_f/2} \]  

(3)

where \( N \) is the total number of fractures with a length \( L \) greater than a constant fracture length \( l \); \( D_f \) is in the range of \([1, 2]\) for 2-D fracture networks; and \( l_{\text{max}} \) is the maximum trace length of fractures in a rock mass.

Differentiating Eq. (3) with respect to \( l \) leads to

\[ -dN = \frac{D_f}{2} l^{D_f/2} l^{-1(D_f/2+1)} dl \]

(4)

The negative sign on the left side of Eq. (4) implies that the number of fractures decreases with increasing trace length. This equation gives the correlation of fracture number and trace length. The total number of fractures \( N_t \) can be calculated by setting \( l = l_{\text{min}} \), which yields

\[ N_t(L \geq l_{\text{min}}) = \left( \frac{l_{\text{max}}}{l_{\text{min}}} \right)^{D_f/2} \]

(5)

where \( l_{\text{min}} \) is the minimum trace length of the fractures in a rock mass. Dividing Eq. (4) by Eq. (5) gives

\[ -\frac{dN}{N_t} = \frac{D_f}{2} l_{\text{min}}^{D_f/2} l^{-1(D_f/2+1)} dl = f(l)dl \]

(6)

where \( f(l) = \frac{D_f}{2} l_{\text{min}}^{D_f/2} l^{-1(D_f/2+1)} \) is the probability density function; according to probability theory, this function satisfies the following equation:

\[ \int_{-\infty}^{\infty} f(l)dl = \int_{l_{\text{min}}}^{l_{\text{max}}} f(l)dl = 1 - \left( \frac{l_{\text{min}}}{l_{\text{max}}} \right)^{D_f/2} \equiv 1 \]

(7)

By eliminating the constant value 1, Eq. (7) becomes

\[ \left( \frac{l_{\text{min}}}{l_{\text{max}}} \right)^{D_f/2} \equiv 0 \]

(8)
Eq. (8) implies that $l_{\text{min}} < l_{\text{max}}$ should be satisfied for Eq. (7) to hold. Eq. (8) is therefore a necessary condition for a fracture distribution to exhibit fractal characteristics. In this study, $l_{\text{min}} / l_{\text{max}} \leq 10^{-3}$ is used as a threshold to enable fluid flow in 2-D fracture networks to be effectively represented using a fractal model. For all of the fractures with trace lengths in the range of $[l_{\text{min}}, l_{\text{max}}]$, the cumulative probability ($R$) can be integrated as

$$R(l) = \int_{l_{\text{min}}}^{l} f(l) \, dl = \int_{l_{\text{min}}}^{l} \frac{D_f}{2} l_{\text{min}}^{D_f/2} l^{-1} \, dl = 1 - \left( \frac{l_{\text{min}}}{l} \right)^{D_f/2}$$  

(9)

Eq. (9) implies that when $l \to l_{\text{min}}$, $R = 0$, and when $l \to l_{\text{max}}$, $R = 1$. As long as $l$ is in the range of $[l_{\text{min}}, l_{\text{max}}]$, the value of $R$ is in the range of $[0, 1]$. Therefore, by assigning random numbers between 0 and 1 to $R$, the correlated trace length $l$ can be back-calculated by

$$l = \frac{l_{\text{min}}}{(1 - R)^{2/D_f}} = \left( \frac{l_{\text{min}}}{l_{\text{max}}} \right) \frac{l_{\text{max}}}{(1 - R)^{2/D_f}}$$  

(10)

To facilitate the calculation, each fracture is labeled with an integer from 0 to $N_t$. For the $i$th fracture, the trace length $l_i$ can be calculated using a random number $R_i$ as follows:

$$l_i = \frac{l_{\text{min}}}{(1 - R_i)^{2/D_f}} = \left( \frac{l_{\text{min}}}{l_{\text{max}}} \right) \frac{l_{\text{max}}}{(1 - R_i)^{2/D_f}}$$  

(11)

where $i = 1, 2, 3, \ldots, N_t$, and $N_t$ is the total number of fractures in the network.

Eq. (11) represents the fractal length distributions in 2-D rock fracture networks and is significantly correlated with the minimum trace length $l_{\text{min}}$, the random number $R$, the fractal dimension $D_f$ and the total number of fractures $N_t$. The validity of Eq. (11) will be verified in Sections 4.2 and 4.3 by comparing the correlation of the fracture number and fracture length with the results of other studies and by comparing the fractal dimension $D_f$ of DFN models generated using Eq. (11) with their theoretical values.
Previous studies of fracture sizes from centimeters to meters at different sites have found that the aperture and the trace length of fractures have positive linear correlations [30-31] or power law correlations [32-33]. Longer fractures usually have higher permeabilities and larger apertures than shorter fractures. In this study, assuming that the DFN models are composed of a large number of small fractures with small hydraulic apertures and a much smaller number of longer fractures with larger apertures, the trace lengths of the fractures are randomly generated according to Eq. (11) and are correlated with the fracture aperture as follows [4, 34]:

$$\frac{\bar{l}^\text{fr}_i - \bar{l}^\text{fr}_\text{min}}{\bar{l}^\text{fr}_\text{max} - \bar{l}^\text{fr}_\text{min}} = \frac{g(e_i) - g(e^\text{min})}{g(e^\text{max}) - g(e^\text{min})}$$ (12)

$$g(e_i) = \text{erf}\left[\frac{\ln e_i - \overline{e}_{\log}}{\sqrt{2}b}\right]$$ (13)

where $\text{erf}$ is an error function; $\overline{e}_{\log}$ and $b$ are the first and second moments of the log-normal distribution of the apertures, respectively; $e^\text{min}$ and $e^\text{max}$ indicate the minimum and maximum apertures, respectively; $e_i$ is the hydraulic aperture of the $i$th fracture; and $l_i$ can be calculated from Eq. (11).

Finally, the aperture of the $i$th fracture can be obtained by forcing $\overline{e}_{\log} = 0$ and $b = 1$ as follows [4, 34]:

$$e_i = \exp\left\{\sqrt{2}\text{erfinv}\left[\frac{\bar{l}^\text{fr}_i - \bar{l}^\text{fr}_\text{min}}{\bar{l}^\text{fr}_\text{max} - \bar{l}^\text{fr}_\text{min}}\left[\text{erf}\left(\frac{\ln e^\text{max}}{\sqrt{2}}\right) - \text{erf}\left(\frac{\ln e^\text{min}}{\sqrt{2}}\right)\right] + \text{erf}\left(\frac{\ln e^\text{min}}{\sqrt{2}}\right)\right]\right\}$$ (14)

where $\text{erfinv}$ is the inverse error function.

Rough surfaces of fractures render the streamlines of fluid flow nonlinear, which can increase the end-to-end distances required for fluid flow through fractures and therefore reduce their equivalent permeability. Here, as shown in Fig. 1, $l$ is the straight length of the fracture pathways in the flow direction, $l_t$ is the tortuous length along the fracture profile, and $e$ is the hydraulic aperture of a single fracture [35]. Generally, $l_t > l$, except for fractures
with flat surfaces (parallel plate model), where \( l_t = l \). The tortuosity is defined as a parameter to depict the ratio of \( l_t \) to \( l \), in order to describe their difference induced by fracture surface roughness in 2-D rock fractures [36]. The correlation of the tortuous length \( l_t \) and the straight length \( l \) of fractures is also considered to be analogous to that of pores in porous media [10, 37], as expressed by

\[
l_t = e^{l - D_T l_t} \tag{15}
\]

where \( D_T \) is the fractal dimension of the non-linear streamline of fluid flow, with a value in the range of [1, 2] for a single fracture in a 2-D rock fracture network. When \( D_T = 1 \), the streamline is linear, resulting in \( l_t = l \), corresponding to a 2-D fracture. As a consequence, the value of \( D_T \) can depict the non-linearity of the streamline, as well as the effect of tortuosity of fluid flow.

3. A fractal model for permeability estimation

3.1 Fluid flow in a single fracture

Although real rock fractures have rough walls and variable apertures, fluid flow through rock fractures is usually described by the cubic law [5], which assumes that fractures consist of two smooth parallel walls. Under these conditions and substituting Eq. (11) and Eq. (15) into the cubic law, the flow rate of the \( i \)th fracture can be written as

\[
q(i) = \frac{e_i^3}{12 \mu} \frac{\Delta P_i}{l_t(i)^{D_T}} = \frac{e_i^{2 + D_T}}{12 \mu} \frac{\Delta P_i}{l_t(i)^{D_T}} = \frac{e_i^{2 + D_T}}{12 \mu} \frac{\Delta P_i}{l_{min}^{D_T}} (1 - R_i)^{2 D_T / D_T} \tag{16}
\]

where \( q(i) \) is the fluid volume through the \( i \)th fracture, \( \Delta P_i \) is the local hydraulic pressure difference applied between the tips of the fracture, and \( \mu \) is the viscosity of the fluid.
If a model consists of a single fracture that satisfies the parallel plate model, then $D_T = 1.0$ and $D_f = 2.0$. Eq. (16) reduces to

$$q_{pp}(i) = \frac{e_i^3}{12\mu l_{min}}(1 - R_i) = \frac{e_i^3}{12\mu l_i} \Delta P$$

(17)

where $q_{pp}(i)$ is the fluid volume through the $i$th fracture based on the parallel plate model. Eq. (17) is the standard form of the cubic law, which is the most simplified case of Eq. (16). The fractal length distribution presented in Eq. (11) is similar to the power law length distribution in 2-D random fracture networks [17-18].

3.2 Fluid flow in a 2-D fracture network

Algorithms for DFN generation and fluid flow through fracture networks were extensively described in the manual of UDEC [38], and only a few principal features are presented here. Three aspects should be addressed to generate a DFN. The first involves the regularization of DFN models. Because fluid flows within connected fracture networks, the fracture elements outside the model, the isolated fracture elements, and the “dead-end” fracture elements do not contribute to the fluid flow and should therefore be deleted. Second, parameters (i.e., trace length, hydraulic aperture, and orientation) should be assigned to each fracture, and the mass continuity equations at each fracture intersection should be established. The third aspect addresses the iteration scheme of these equations for given boundary conditions. Steady-state fluid flow was adopted in this study for calculating the equivalent permeability of DFNs, and a generic hydraulic boundary condition with a constant hydraulic gradient in the $x$-direction was assumed, indicating that the directivity of the equivalent permeability is horizontal (see Fig. 2).
In a 2-D fracture network, fractures are line segments, and fracture intersection points are denoted as nodes. The summation of the flow rate of each fracture connected to a node is zero or equals the added source term. Similarly, the summation of the total flow rate of the entire fracture network equals zero or the summation of the source terms added at each node. Fig. 2(b) shows a region composed of a node $m$ and several fractures from the DFN model shown in Fig. 2(a). Taking into account the balance of fluid flow, the equilibrium equation of node $m$ can be written as

$$\left(\sum_{n=1}^{M} q_n\right)_m + Q_m = 0, (m = 1, 2, \ldots, N_{node})$$

where $q_n$ is the fluid volume through the fracture element $n$, $M$ is the total number of fracture elements connected to node $m$, $Q_m$ is a source term at node $m$, and $N_{node}$ is the total number of nodes in the entire fracture network. Generally, under steady-state flow without any source terms, $Q_m = 0$ and $M \leq 4$ (two fractures intersect at one node) given that the probability that more than two fractures intersect at the same node is nearly zero. Steady-state flow was assumed; therefore, the fluid volume that flows into a fracture network is equal to that flowing out of the network. The fluid rate at the outlet boundary was then utilized to estimate the equivalent permeability of the DFN model.

Using Darcy’s law as shown in Eq. (19), the equivalent permeability $K$ is obtained in Eq. (20):

$$Q = A \frac{K \Delta P'}{\mu L'}$$

$$K = \frac{\mu L'}{A \Delta P'} Q(e, D_f, l_{min}, D_j, R, \Delta P')$$

where $A$ is the cross-sectional area perpendicular to the fluid flow direction, $L'$ is the length of the fracture network in the fluid flow direction, $Q$ is the total fluid volume through the fracture network per second, $\Delta P'$ is
the hydraulic pressure difference of the DFN model between the inlet boundary and the outlet boundary, and $K$ is the equivalent permeability of the fracture network.

Eqs. (16) and (20) are the fundamental equations of the proposed fractal model for calculating the equivalent permeability of fracture networks and are a function of $e, D_f, l_{\text{min}}, D_T, R$ and $\Delta P'$. In this model, $D_f$ can be calculated by applying the box-counting method to fracture networks [39] and $D_T$ is a given input parameter based on in situ geological survey data. No empirical constants are involved in this fractal model, and all of the parameters have clear physical meanings.

3.3 Basic parameters

Baghbanan and Jing [4] studied the hydraulic properties of DFNs based on field mapping results of a real rock mass. For simplification, we used the values of most of the parameters from their study. The maximum trace length $l_{\text{max}}$ and minimum trace length $l_{\text{min}}$ were 500 m and 0.5 m, respectively, with a ratio $l_{\text{min}}/l_{\text{max}}$ of 0.001, which satisfied the condition of Eq. (8). The range of $D_f$ was 1.3 – 1.8, below which the connectivity of the fracture networks becomes very poor and above which the rock masses are so fragmented that they rarely exist in the nature. Wakabayashi et al. [40], based on the 10 classical single fractures suggested by Barton [41], studied the relation of $D_T$ and JRC (joint roughness coefficient) and derived a regression equation, where the JRC is a commonly used parameter for describing the surface roughness of rock fractures [42]. Their results show that the range of $D_T$ was 1.000 – 1.018, which roughly corresponds to JRC = 0 – 20. The maximum and minimum apertures were 100 $\mu$m and 1 $\mu$m, respectively. These parameters were utilized to generate the fracture networks with the Monte Carlo method using a fractal probability density function for the trace length distribution.
according to Eq. (11). The fracture orientation and fracture center point distribution were assumed to be uniformly distributed because this study is only focused on the validity of the proposed fractal model and its influence on the equivalent permeability of 2-D fracture networks.

4. Results and analyses

4.1 Determination of the model size

The size of a DFN model should be determined prior to performing the fluid flow simulation. In Eqs. (3), (16) and (20), the model size was unknown. It is well known that \( D_f \) is not correlated with the model size, which is a fundamental property of fractal models. For fracture networks, the fractal dimension is positively correlated with the mass density (total length per square meter) of the fractures. To obtain the mathematical relation between \( D_f \) and the mass density \( d_m \), a series of DFN models with different values of \( d_m \) were generated, and their fractal dimensions \( D_f \) were calculated using the box-counting method. The results are shown in Fig. 3, in which a regression function could be obtained as follows:

\[
D_f = a'' + b'' \ln(d_m + c'')
\]  

(21)

where \( a'' \), \( b'' \), and \( c'' \) are three regression parameters equal to 0.4594, 0.2797 and 8.4968, respectively.

According to the definition of the mass density of fractures, \( d_m \) can be expressed as follows:

\[
d_m = \frac{\sum l_i}{N_t L_t} = \frac{\sum l_i}{(L_t)^2}
\]  

(22)

where \( L_t \) is the theoretical model length.

Substituting Eq. (22) into Eq. (21), \( L_t \) can be calculated as follows:
Using the calculated value of $L_t$ in Eq. (23), the number density of fractures $d_n$, which defines the total number of fractures in a given area of the fracture network, is obtained as follows:

$$
d_n = \frac{N_i}{L_t^2} \tag{24}
$$

Consequently, for a constant fractal dimension $D_f$, the mass density $d_m$ and the number density $d_n$ are also constants. For an arbitrary model length $L_n > L_t$, the new fracture number $N'_t$ can be calculated based on Eqs. (23) and (24) as follows:

$$
N'_t = \frac{N_i L_n^2}{L_t^2} = N_i L_n^2 \left[ \exp \left( \frac{D_f - a'''}{b'''} \right) - c''' \right] \left/ \sum_{j=1}^{N_i} l_j \right. \tag{25}
$$

Considering previous studies on the representative elementary volume (REV) of fracture networks [4], we chose a model size $L_n$ of 5.0 m for the calculations. The mass density $d_m$, the theoretical fracture number $N_i$, the theoretical model size $L_n$, and the fracture number $N'_t$ of any model can be calculated with Eqs. (22), (5), (23) and (25), respectively, as shown in Table 2.

### 4.2 Validity of the fractal length distribution

The length of each fracture was generated using Eq. (11) until the index $i$ reached $N_i'$ for different fractal dimensions $D_f$ from 1.3 to 1.8 (Table 2). Higher values of the fractal dimension $D_f$ lead to higher probabilities of generating longer fractures. The fracture number $n(l, 0.5)$ represents the number of fractures with lengths in the range of $(l - 0.5, l + 0.5)$. Fig. 4 and Table 3 show the statistical results of the relation between fracture length and fracture number using the parameters shown in Table 2. For each $D_f$, three sets of random number seeds were
utilized to generate different fracture length distributions using Eq. (11). Each fracture length-fracture number curve was fitted using a power law function in the form of

\[ n(l, 0.5) = al^a \]  

where \( n(l, 0.5) \) is the number of fractures with lengths in the range of \((l - 0.5, l + 0.5)\), \(a\) is the coefficient of proportionality, and \(a\) is the power law exponent.

With increasing \(D_f\), more fractures with relatively long lengths appear, and the length of the longest fracture increases, as shown in Fig. 4. Table 3 and Fig. 5 show that the average power law exponent \(a\) ranges from 2.29 to 1.70, corresponding to fractal dimensions \(D_f\) of 1.3 to 1.8. These parameters have a linear relationship that can be expressed by

\[ a = -D_f + 3.49 \]  

For 2-D fracture networks, \(D_f\) ranges from 1 to 2, which results in values of the power law exponent \(a\) from 2.49 to 1.49 according to Eq. (27). These results reveal that the fracture length also follows a power law length distribution with the power law exponent \(a\) linearly correlated with the fractal dimension \(D_f\). The theoretical value of the power law exponent \(a\) ranges from 1.49 to 2.49, which agrees well with the values reported in the literature from in situ measurements and theoretical analyses (Table 4) [17-18, 43-45]. Therefore, the proposed fractal length distribution approach is reliable and is capable of describing the characteristics of fracture distributions using a fractal dimension method.

4.3 Fractal evaluation methods
The stochastic DFN modeling technique was utilized to generate and characterize fracture networks. Measuring length density is difficult because each model contains a large number of fractures. The geometric distribution of rock fractures in a fracture network has fractal characteristics [25]. Therefore, the fractal dimension may serve as an effective approach for assessing the geometric characteristics of rock fractures.

Because it is a simple and precise approach, the box-counting method was used and improved in this study to calculate the fractal dimensions of the DFN models. First, the original image was converted to monochrome (Fig. 6(a)), and a binary image (Fig. 6(b)) was then obtained based on the adaptive threshold of the grayscale. The example shown in Fig. 6(b) is 280 pixels in length and 280 pixels in width, and each pixel has a value of either 1 or 0 (null). Second, the image was covered by square boxes with different dimensions from 280x280 pixels (1 box) to 1x1 pixel (280x280 boxes). If the pixels in a box were not null, the box was given a value of 1; otherwise, it was given a value of 0. Finally, a log-log plot of the box count ($N_c$) vs. the number ($N_b$) of total boxes (Fig. 6(c)) was drawn, and the slope represents the fractal dimension. To verify the validity of this program, a series of well-known fractal graphs were generated, and the results were compared with the theoretical values. The deviations between the calculated and theoretical values are less than 2%; therefore, the program is considered to reliably estimate the fractal dimensions $D_f$.

For each fractal dimension $D_f$, 10 sets of DFN models (numbered from 1 to 10) were developed using the fractal length distribution of Eq. (11). The fractal dimensions ($D_f$) of these DFN models were then calculated using the box-counting method and compared with their theoretical values (Fig. 7). When $D_f$ was small (e.g., 1.3 and 1.4), the calculated fractal dimensions $D_f$ were underestimated compared to their theoretical values because many
isolated fractures that did not connect with other fractures were deleted from the models. At higher values of $D_f$ (1.5 and 1.6), the calculated and theoretical values agreed with each other, and the discrepancies decreased because the fractures became denser and the connectivity of the models improved. These results are robust evidence of the validity of applying Eq. (3) from porous media to fractured rock masses and the validity of the fractal length distribution presented in Eq. (11).

4.4 Characteristics of the flow patterns

Several examples of DFN models with side lengths of 5.0 m and $D_f$ values from 1.3 to 1.6 are shown in the left column of Fig. 8. These models were generated using the Monte Carlo method based on the parameters presented in Section 3. In these models, the upper and lower boundaries were assumed to be impermeable, and fluid flowed from left to right (see Fig. 2) with a hydraulic head of 5 m. When the inflow volume was equal to that of the outflow, the fluid flow was regarded as a steady-state flow. The flow rate distributions at these conditions are shown in the central column of Fig. 8. The right column of Fig. 8 shows corresponding remarks. In Figs. 8(b) and 8(e), preferential flow paths exist along the relatively long fractures that are subparallel to the flow direction (horizontal) and particularly in those that intersect the inlet and outlet boundaries. With an increase of $D_f$ (see Figs. 8(h) and 8(k)), more short fractures achieve large flow rates and the flow rate distributions within the networks become more homogeneous. These observations were also reported by De Dreuzy et al. [17-19], who found that the connectivity and permeability of 2-D fracture networks are controlled by the largest fracture in the system when only a few fractures exist and controlled by the fractures that are smaller than average when a large number of fractures exist.
4.5 Influences of $D_f$ and $D_T$ on the equivalent permeability

As shown in Eq. (16), the proposed model includes a random number $R_i$ that introduces randomness to the fracture networks. Ten random numbers corresponding to each fractal dimension $D_f$ were generated and applied to the DFN modeling. The relation between the equivalent permeability $K$ and the value of $D_f$ calculated utilizing these random numbers is shown in Fig. 9. Due to the randomness of the trace length and the location and orientation of rock fractures induced by the random numbers, DFN models with the same $D_f$ value can have large differences in their geometric distributions, which results in variations in the calculated equivalent permeabilities (see Fig. 9) [19]. When $D_f$ is small (e.g., 1.3), the equivalent permeability varies by approximately 5 orders of magnitude. The range of variation decreases to less than 2 orders of magnitude when $D_f$ becomes large (e.g., 1.7-1.8). With increasing $D_f$, the influence of the random number decreases, which is reasonable because the permeability of a model with only a few fractures can be affected more by the randomness of the trace length, location and orientation of the fractures than models that contain many fractures (see Figs. 8(a) and 8(j)). By taking the mean value of the results for each $D_f$, we obtained an approximate curve that indicates that the changes in equivalent permeability with $D_f$ follows an exponential law.

Fig. 10 shows the relation between the equivalent permeability and $D_T$ with trace length ratios $l_{min}/l_{max}$ of $2.0\times10^{-4}$, $1.0\times10^{-4}$ and $0.5\times10^{-4}$, which were obtained using a constant $l_{min}=0.05$ m and $l_{max}$ values of 250 m, 500 m and 1000 m, respectively. With an increase of $D_T$, the equivalent permeability decreases because higher values of $D_T$ represent greater tortuosity and thus greater resistance to fluid flow in the fractures, which results in a lower equivalent permeability. Fig. 10 also shows that as the trace length ratio $l_{min}/l_{max}$ increases, the equivalent
permeability decreases. The increase of $l_{\text{min}}/l_{\text{max}}$ will decrease the total number of fractures $N_t$ for a given $D_f$ of a DFN model. The smaller total number of fractures $N_t$ results in worse connectivity of the network and decreases the equivalent permeability in turn. The three regression functions in Fig. 10 show that the changes in equivalent permeability with $D_T$ follow exponential laws, which are affected by the trace length ratio.

The cubic law in the form of Eq. (17) is usually utilized for fluid flow in fractures by assuming that each fracture consists of two parallel walls. According to Eq. (16), the modified cubic law that considers the effect of $D_T$ can be rewritten as

$$q(i) = \frac{e^{2+D_T} \cdot AP}{12 \mu l^{1/2}_i}$$

$$D_T$$ is the source of the difference between Eq. (17) and Eq. (28), and its effects on DFN models can be estimated by the equivalent relative error $\bar{\delta}$:

$$\bar{\delta} = \frac{Q_{\text{cubic}} - Q_{\text{tortuosity}}}{Q_{\text{cubic}}}$$

where $Q_{\text{cubic}}$ is the fluid volume through the DFN model based on the parallel plate model (see Eq. (17)) and $Q_{\text{tortuosity}}$ is the fluid volume considering the tortuosity of the flow (see Eq. (28)).

Fig. 11 shows the relation between $\bar{\delta}$ and $D_T$ at different values of $D_f$. The equivalent relative error $\bar{\delta}$ increases with an increase in $D_T$ and follows an exponential law. The values of $\bar{\delta}$ vary little with different random numbers at a given $D_f$, which indicates that the random number mainly represents the macro-scale properties of the fracture network. The four figures in Fig. 11 show that the values of $\bar{\delta}$ vary little among cases with different values of $D_f$, which shows that $D_f$ is another macro-scale parameter that, along with the random number, determines the geometric characteristics of the fracture networks. In the cases with $D_T = 1.018$, which
corresponds to a JRC value of 20, the maximum values of \( \bar{\delta} \) range from 17.64% to 19.51%, which indicates that the effect of tortuosity is not negligible and should be included in fractal models to accurately estimate the hydraulic behavior of fracture networks.

### 4 Conclusions

In this study, a fractal model was established to assess the equivalent permeability of 2-D rock fracture networks. The fractal dimension \( D_T \) and the fractal dimension \( D_f \) were used in the model to represent the effects of the tortuosity of fluid flow in the fractures (micro-scale) and the geometric characteristics of the fracture distributions (macro-scale), respectively. Fluid flow was simulated in the generated fracture network models, and the relation between the fractal dimension and the equivalent permeability was analyzed.

The results showed that the correlation of fracture number and fracture length based on the proposed fractal length distribution in this study agrees well with reported values from the literature, which confirmed the reliability of the proposed length distribution approach. Comparisons of the fractal dimension \( D_f \) between the values calculated using the box-counting method and the theoretical values agree with each other, which verified the validity of the fractal DFN models that were developed using the proposed fractal length distribution. When \( D_f \) is small (e.g., less than 1.5), fluid flow mainly occurs in a few long fractures that are subparallel to the flow direction and particularly in the fractures that intersect the inlet and outlet boundaries of the models. When \( D_f \) exceeds a certain value (e.g., 1.5), the flow rate distribution becomes more homogeneous, and shorter non-persistent fractures dominate the preferential flow paths. The equivalent permeabilities of models generated using different random numbers vary significantly with changes of \( D_f \) when \( D_f \) is small (e.g., less than 1.5), and
they become more stable when \( D_f \) is relatively large (e.g., greater than 1.6). This behavior is consistent with the observations of the flow paths, which show that the models become more homogeneous at larger values of \( D_f \). Therefore, a mathematical expression between the equivalent permeability \( K \) and the fractal dimension \( D_f \) (e.g., the exponential relationship presented in this study) can be expected to be applicable for models with large values of \( D_f \). For models with small values of \( D_f \), other parameters, such as connectivity, should be taken into account to improve the accuracy of the predictions. Compared with the parallel plate model, the maximum deviation of the calculated flow volume that considers the effect of tortuosity \( (D_T) \) can be as high as 19.51\% when \( D_T = 1.018 \), which corresponds to a JRC value of 20. These results show that both the geometric characteristics of the fracture distributions and the geometric characteristics (surface roughness) of single rock fractures (the source of tortuosity) have significant influence on the hydraulic behavior of fracture networks. Further development of the proposed model is required to estimate its scaling effects, which might have important impacts on the equivalent permeability and were not considered in this study.

**Acknowledgements:**

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**References:**


[38] Itasca Consulting Group Inc. UDEC User’s guide, ver 4.0, Minneapolis, Minnesota, 2004.


Fig. 1 Schematic view of fluid flow through a rough fracture. Tortuosity induced by fracture surface roughness makes the actual traveling length $l_t$ of fluid flow larger than the straight length $l$. 
Fig. 2 (a) Hydraulic boundary conditions applied to a fracture network. The quadrilaterals represent hydraulic pressures, and the hydraulic gradient was fixed at 1 kPa/m to generate horizontal fluid flow. (b) Fluid flow equilibrium of a node in a fracture network.
Fig. 3 The relation between the fractal dimension $D_f$ and the mass density of fractures.

$$y = 0.4594 + 0.2797 \ln(x + 8.4968)$$
Fig. 4 Correlation of fracture number and fracture length with fractal dimensions $D_f$ ranging from 1.3 to 1.8.
Fig. 5 Linear relation between the power law exponent $a$ and the fractal dimension $D_f$. 

$y = -x + 3.49$

$R^2 = 0.8378$
Fig. 6 Image processing and calculation of the fractal dimension.
Fig. 7 Comparisons of fractal dimension $D_f$ between calculated values and theoretical values using 10 sets of DFN models for each $D_f$. 
Fig. 8 Geometric distributions of DFN models (a, d, g, j; left column), correlated flow rate distributions and flow paths (b, e, h, k; central column) and remarks (c, f, i, l; right column) with varying fractal dimensions, $D_f$, from 1.3 to 1.6.
Fig. 9 Equivalent permeabilities of models with $D_f$ values from 1.3 to 1.8 generated using 10 sets of random numbers.
Fig. 10 Relations between equivalent permeability with the fractal dimensions $D_T$ and the trace length ratios $l_{min}/l_{max}$. 

$y = \exp(-7.5x-4.0)$  
$R^2=1$

$y = \exp(-7.5x-4.6)$  
$R^2=1$

$y = \exp(-7.5x-5.2)$  
$R^2=1$
Fig. 11 Relative errors of DFN models with $D_T$ values from 1.000 to 1.018 and $D_f$ values from 1.3 to 1.6.
Table 1 Review of fractal DFN models used to calculate the permeability of rock masses.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Year</th>
<th>Purpose of the model</th>
<th>Outcome of the study</th>
</tr>
</thead>
<tbody>
<tr>
<td>De Dreuzy et al.</td>
<td>2001a; 2001b; 2002</td>
<td>Investigation of the hydraulic properties of 2-D fracture networks with random fracture geometries that follow a power law length distribution.</td>
<td>They analyzed the influence of the power law exponent $a$ in their models and found that if $a$ was greater than 3, the classical percolation model based on a population of small fractures was applicable, and the fluid flow appeared to be relatively homogeneous in the flow direction. In contrast, if $a$ was less than 2, the applicable model was made up of the largest fractures of the network, and the main flow paths were composed of a few large fractures. Between the two limits ($2 &lt; a &lt; 3$), relatively uniform fluid flow occurred in all of the fractures.</td>
</tr>
<tr>
<td>Yu et al.</td>
<td>2002</td>
<td>Development of a fractal model to calculate the equivalent permeability of bi-dispersed porous media.</td>
<td>They extensively evaluated the influences of the fractal dimension $D_f$ (which represents the fracture distribution) and $D_T$ (which represents tortuosity) on the equivalent permeability. They found that the equivalent permeability was a function of the tortuous fractal dimension, pore area fractal dimension, sizes of particles and clusters, micro-porosity inside clusters, and the effective porosity of a medium.</td>
</tr>
<tr>
<td>Yu et al.</td>
<td>2005</td>
<td>Establishment of a 2-D fractal model to calculate the permeability of a porous media model generated by the Monte Carlo method.</td>
<td>Their model can predict the transport properties (i.e., permeability, thermal conductivity, dispersion coefficient and electrical conductivity) of saturated or unsaturated fractal porous media.</td>
</tr>
<tr>
<td>Xu et al.</td>
<td>2006</td>
<td>Development of a fractal model for fluid flow in porous media that are</td>
<td>They found that the permeability of the model that incorporated the flow tortuosity was</td>
</tr>
<tr>
<td>Reference</td>
<td>Year</td>
<td>Description</td>
<td>Result/Methodology</td>
</tr>
<tr>
<td>-----------</td>
<td>------</td>
<td>-------------</td>
<td>--------------------</td>
</tr>
<tr>
<td>Bejan and Lorente [22] and Bejan and Zane [23].</td>
<td></td>
<td>embedded with randomly distributed fractal-like tree networks using the constructal theory proposed by Bejan and Lorente [22] and Bejan and Zane [23].</td>
<td>approximately 20% lower than the model that did not consider the tortuosity.</td>
</tr>
<tr>
<td>Zou et al. [24]</td>
<td>2007</td>
<td>Establishment of a fractal model to analyze the 3-D surfaces of rock fractures.</td>
<td>Their results indicated that larger values of the fractal dimension $D$ of the profile of a rough surface or smaller values of the scaling constant $G$ signify a smoother surface topography.</td>
</tr>
<tr>
<td>Jafari and Babadagli [25]</td>
<td>2012</td>
<td>Estimation of the equivalent permeability of fracture networks using numerical simulations</td>
<td>They derived a nonlinear multivariable regression to address the equivalent permeability by calculating three parameters: $X1$ (the connectivity index), $X2$ (the box-counting fractal dimension of the fracture lines) and $X3$ (the hydraulic conductivity).</td>
</tr>
<tr>
<td>Zheng el at. [26]</td>
<td>2012</td>
<td>Development of a fractal model for fluid flow in porous media that are embedded with randomly distributed fractal-like tree networks using the constructal theory proposed by Bejan and Lorente [22] and Bejan and Zane [23].</td>
<td>They derived an analytical expression for the gas permeability in dual-porosity media based on the pore size of the matrix and the diameter of the mother channel of embedded fractal-like tree networks. They found that for a certain fracture network, the dimensionless permeability $K^*$, which is defined as the ratio of the porous matrix permeability $K_m$ to the fracture network permeability $K_f$, increases with increasing matrix porosity $e$.</td>
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Table 2 Parameters associated with a model size of 5.0 m×5.0 m.

<table>
<thead>
<tr>
<th>$D_f$</th>
<th>$d_m$ (m/m$^2$)</th>
<th>$L_t$ (m)</th>
<th>$N_t$</th>
<th>$L_n$ (m)</th>
<th>$N_t'$</th>
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<tr>
<td>1.3</td>
<td>11.6905</td>
<td>2.8276</td>
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<td>5.0</td>
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<tr>
<td>1.4</td>
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<tr>
<td>1.5</td>
<td>32.7688</td>
<td>2.7023</td>
<td>177</td>
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<td>1.6</td>
<td>50.5020</td>
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<tr>
<td>1.7</td>
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<td>5.0</td>
<td>1200</td>
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<tr>
<td>1.8</td>
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<td>2.8814</td>
<td>501</td>
<td>5.0</td>
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Table 3 Calculation results based on the proposed fracture length distribution (Eq. (11)).

<table>
<thead>
<tr>
<th>$D_f$</th>
<th>Case_1</th>
<th>Case_2</th>
<th>Case_3</th>
<th>Case_1</th>
<th>Case_2</th>
<th>Case_3</th>
<th>Case_1</th>
<th>Case_2</th>
<th>Case_3</th>
<th>Case_1</th>
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<tbody>
<tr>
<td></td>
<td>Value</td>
<td>Average</td>
<td>Value</td>
<td>Average</td>
<td>Value</td>
<td>Average</td>
<td>Value</td>
<td>Average</td>
<td>Value</td>
<td>Value</td>
<td>Average</td>
<td>Average</td>
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<tr>
<td>1.3</td>
<td>155.41</td>
<td>-2.3231</td>
<td>157.12</td>
<td>-2.3820</td>
<td>123.22</td>
<td>-2.1613</td>
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<td>1.4</td>
<td>129.65</td>
<td>-1.8506</td>
<td>127.05</td>
<td>-1.9661</td>
<td>158.78</td>
<td>-2.0876</td>
<td>0.7713</td>
<td>0.7851</td>
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<td>1.5</td>
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<td>-1.8478</td>
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<td>-1.9599</td>
<td>268.89</td>
<td>-2.1298</td>
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<td>0.9073</td>
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<td>1.6</td>
<td>263.01</td>
<td>-1.9245</td>
<td>299.36</td>
<td>-1.9525</td>
<td>235.04</td>
<td>-1.7109</td>
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<td>0.8774</td>
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<td>-1.6881</td>
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<td>0.7285</td>
<td>0.7360</td>
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Table 4 Comparisons between the values of the power law exponent $a$ in this and other studies.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Year</th>
<th>Value of the power law exponent $a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dverstop and Anderson</td>
<td>1989</td>
<td>1.7</td>
</tr>
<tr>
<td>Tsang et al.</td>
<td>1996</td>
<td>3.0</td>
</tr>
<tr>
<td>Bour and Davy</td>
<td>1997</td>
<td>1.0 – 3.0</td>
</tr>
<tr>
<td>De Dreuzy</td>
<td>2001</td>
<td>0.0 – 3.5</td>
</tr>
<tr>
<td>This study</td>
<td></td>
<td>1.49 – 2.49</td>
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