A Dynamic Bond Pricing Model with Application to the Japanese Government Bonds

Kenji Kamizono *, Takeaki Kariya ̈ and Yoshiro Yamamura ̉

Abstract

In this paper, we generalize the cross-sectional fixed-coupon bond pricing model of Kariya et. al. ̈ to a dynamic one. The bond prices are modeled as the present values of the future cash-flows where the discount functions are stochastic and may depend on the bond attributes. In our framework, the cross-sectional and time-series covariance structure among the stochastic discount functions depends on the difference of the time-to-maturity of the bonds. We also propose a bond price forecast method using our model. The empirical result and the forecast performance on the Japanese government bonds are presented.

Key words: Fixed-coupon bond pricing model. Japanese government bonds. Generalized least squares.
JEL classification C üúɼ, C üÿɼ, G øùɼ, G øþõ

1 Introduction

Early in ̈, Kariya ̈ proposed a statistical approach to a bond pricing model. There, the bond price was modeled as the present value of the future cash-flows for which the discount functions, in general, are stochastic and attribute dependent. Kariya and Tsuda ̈, ̈ then demonstrated that this model was empirically effective for pricing Japanese government bonds. Recently, Kariya et. al. ̈claried the theoretical relation between this model and the traditional spot rate approach, and also proposed a specic formulation with a polynomial mean discount function as well as a cross-sectional correlation structure depending on the difference of the time-to-maturity of the bonds. Their model was cross-sectional and one of the remaining issues there was generalization to a dynamic model which can incorporate time-series correla-

*Faculty of Economics, Nagasaki University, Katafuchi, Nagasaki-shi, Nagasaki ₄₄₀₄₄, Japan. e-mail k-kamiz@nagasaki-u.ac.jp

Graduate School of International Administration, Josai International University, Chiyoda-ku, Tokyo ₄₃₂₃₄, Japan. e-mail thekariya@gmail.com

Graduate School of Global Business, Meiji University, Kanda-Surugadai, Chiyoda-ku, Tokyo ₄₃₃₃₄, Japan. e-mail yyama@kisc.meiji.ac.jp
tion among the bond prices.

One of the main features of the cross-sectional model of the covariance structure of the stochastic discount functions. There, the discount functions of different bonds may be different, which means that the discount function used for discounting the cash-flows of a single bond may be different from those used for discounting the cash-flows of other bonds. Each of these attribute-dependent discount functions consists of a deterministic part and a stochastic part. The stochastic part of the discount functions of two different bonds may be correlated with each other. The correlation structure specified by is such that the correlation between the discount functions of two bonds is higher as the time-to-maturity of these bonds is closer to each other. The correlation decays exponentially as the difference of the time-to-maturity of the bonds is larger.

In this paper, we generalize this specification of the cross-sectional covariance structure of the stochastic discount functions to a dynamic one. We adopt a similar covariance structure between the discount functions viewed from time \( t \) and \( t' \), for each \( t \). That is, the correlation between two discount functions viewed from these two different time points is large as the time-to-maturity of the bonds is closer to each other, and the correlation decays exponentially as the difference of the time-to-maturity is larger. With such a dynamic covariance structure in the stochastic discount functions, the bond pricing model takes the form of a seemingly unrelated regression (SUR) model, where we have one regression model for each time point and these regression models are correlated to each other through the stochastic part of the discount functions. In such a situation, although each regression model can be estimated separately as in the cross-sectional model of \( \Gamma \), it is more efficient to consider the dynamic correlation between the cross-sectional regression. However, if the number of the bonds at each time point is large, as is the case of Japanese government bonds, then it is difficult to perform numerical calculation. Therefore, we will propose to estimate the model by using the data of adjacent three time points at one time. Our model can be used to forecast the future bond prices. First, the estimated regression coefficients form time-series data by which we can forecast the mean discount function in future. Second, by using the estimated dynamic covariance structure as well as the regression residual of the most recent time point, we can forecast the stochastic discount function in future. Then, by combining these two components, we can forecast the bond price in future. In this paper, we will investigate the forecast performance for the Japanese government bonds.

This paper is organized as follows. In Section \( \Gamma \), we describe the model specification. The estimation method of this model will be presented in Section \( \Delta \). Section \( \Gamma \) will describe the Japanese government bond price data we have used and then present the model performance and the estimation result. Finally, we will explain the forecast method and the forecast performance for the Japanese government bonds in Section \( \Xi \).
2 Model Specification

In this section, we will define our government bond pricing model. Suppose that for each time $t$, we have $G_t$ non-defaultable government bonds bearing fixed coupons. For each $0 \leq g \leq G_t$, let

$$s_{gt} \triangleq \begin{cases} \text{accrued interest} = s_{gt}^f & \text{if } j = \emptyset \\ \emptyset & \text{if } \emptyset < j < M_{gt} \\ \emptyset^+ \emptyset^- & \text{if } j = M_{gt} \end{cases}$$

be the sequence of future time points, viewed from time $t$, on which the coupon and principal payments of the bond $g$ are supposed to be paid. By denition, $s_{gt,M_{gt}}$ is the time-to-maturity of the bond $g$ from time $t$. Let $c_{gtj}$ be the amount of payment paid at time point $s_{gtj}$ in future from time $t$. If the interest payments are to be paid semi-annually with a coupon rate $c$ and if the face value of the bond is $\emptyset$, as is the case for the Japanese government bonds, then

$$c_{gtj} = \begin{cases} \emptyset & \text{if } j = \emptyset \\ \emptyset & \text{if } \emptyset < j < M_{gt} \\ \emptyset & \text{if } j = M_{gt} \end{cases}$$

The bond price $P_{gt}$, after subtracting the accrued interest paid from the buyer to the seller, is modeled as the present value of the cash-ows of the bond as

$$P_{gt} = \prod_{j=1}^{M_{gt}} C_{gtj} D_{gtj}$$

Here, $D_{gtj}$ denotes the discount function of the bond $g$ for the cash-ow occurring at $s_{gtj}$-period future from time $t$. Notice that $D_{gtj}$ may depend on $g$, which means that the discount functions may be different for different bonds. We assume that $D_{gtj}$ can be written as

$$D_{gtj} = \overline{D}_{gtj} + \emptyset_{gtj}$$

$$\overline{D}_{gtj} = \prod_{j=1}^{M_{gt}} \exp \{- \emptyset_{gth} \}$$

Here, the $\emptyset_{gth}$ are unknown parameters and the $\emptyset_{gtj}$ are random variables with expectation $\emptyset$ and

$$\text{Cov} \emptyset_{gth} \emptyset_{hth} = \emptyset_{gth} \emptyset_{hth}$$

where

$$\emptyset_{gth} = \begin{cases} \emptyset & \text{if } g = h \\ \emptyset \exp \{- \emptyset_{gth} \} & \text{if } g \neq h \end{cases}$$

In a more general framework, $\overline{D}_{gtj}$ may depend on the bond attribute as in $\emptyset$. In this paper, however, we assume for simplicity that the $\emptyset_{gth}$, and thus the $\overline{D}_{gtj}$, are common for all bonds.

In addition to the cross-sectional covariance structure given in the above, we assume that
the \( g_{ij} \) have a time-series covariance structure which can be written as

\[
\text{Cov} \begin{bmatrix} g_{ij} & g_{ik} \end{bmatrix} = \begin{bmatrix} \sigma_i \sigma_j & \rho_{ij} \sigma_i \sigma_j \end{bmatrix} \exp \left\{ -\frac{\left| \sigma_i \sigma_j - \rho_{ij} \sigma_i \sigma_j \right|}{\lambda} \right\}
\]

and

\[
\text{Cov} \begin{bmatrix} g_{ij} & g_{jk} \end{bmatrix} = 0 \quad \text{for} \quad |t - u| \geq \Delta.
\]

In the covariance structure \( \cdots \), we assume that \( \Delta > 0 \), \( 0 \leq \sigma_i \leq \sigma \), \( \sigma_i \geq 0 \), \(- \sigma \leq \rho_{ij} \leq \sigma\), \( \sigma_i \geq 0 \) and \( \rho_{ij} \geq 0 \). As in \( \cdots \), the mean discount function is a polynomial function and covariance structure reflects the difference of the maturities and the coupon payment dates. Let

\[
y_t = \begin{bmatrix} y_{gt} \\ \vdots \\ y_{G_G, t} \end{bmatrix}, \quad X_t = \begin{bmatrix} X_{G_G} & X_{G_{gt}} \\ \vdots & \vdots \\ X_{G_G, t} & X_{G_{G_G, t}} \end{bmatrix}, \quad \Omega_t = \begin{bmatrix} \Omega_{G} \\ \vdots \\ \Omega_{G_G, t} \end{bmatrix},
\]

where

\[
y_{gt} = P_{gt} \cdot \sum_{j=1}^{M} C_{g_{ij}}
\]

\[
X_{g_{kt}} = \sum_{j=1}^{M} C_{g_{ij}} s_{g_{ij}}
\]

\[
\Omega_{gt} = \sum_{j=1}^{M} C_{g_{ij}} \Omega_{g_{ij}}
\]

Then, for each time \( t \), we obtain the multiple regression model

\[
y_t = X_t \Omega_{G} + \Omega_t.
\]

The covariance matrix of \( \Omega_t \) can be written as

\[
\text{Cov} \begin{bmatrix} \Omega_t \end{bmatrix} = \begin{bmatrix} \Omega_t \end{bmatrix} \Omega_t \Omega_t
\]

where

\[
\begin{bmatrix} \Omega_t \end{bmatrix} \Omega_t = \begin{bmatrix} \Omega_{t G} \end{bmatrix} \Omega_{t G}
\]

\[
\Omega_{t G} = \sum_{j=1}^{M} C_{g_{ij}} \Omega_{g_{ij}}
\]

On the other hand, according to our time-series covariance structure \( \cdots \) of the stochastic discount functions, the error-terms \( \Omega_t \) and \( \Omega_t \) are correlated to each other as

\[
\text{Cov} \begin{bmatrix} \Omega_t \end{bmatrix} \begin{bmatrix} \Omega_t \end{bmatrix} = \begin{bmatrix} \Omega_t \end{bmatrix} \Omega_t \Omega_t
\]

where
3 Estimation Method

Suppose that the bond prices are observed at each time \( t = 1, \ldots, T \). Then we have a cross-sectional bond pricing model \( P_t \) for each \( t \), where the regression models are correlated to each other through their error terms \( \epsilon_t \). Such a model is called a seemingly unrelated regression (SUR) model, and it is known that although each regression coefficient \( \beta_t \) can be estimated separately from the cross-sectional regression, it is more efficient to estimate the \( \beta_t \) simultaneously using the covariance structure among the error terms \( \epsilon_t \). However, the number of bonds at each time \( t \) may often be too large to perform numerical calculation. For instance, it is typically more than \( n \) for the case of Japanese government bonds. In such a case, it takes too much time to estimate the parameters if we pool all the data through the sample period. Therefore, in this paper, we shall propose to estimate the parameters at each time \( t \) by using the adjacent three times \( t-1, t, \) and \( t+1 \). For this purpose, write

\[
\tilde{Y}_t = \begin{bmatrix} y_{t-1} \\ y_t \\ y_{t+1} \end{bmatrix}, \quad \tilde{X}_t = \begin{bmatrix} X_{t-1} & O & O \\ O & X_t & O \\ O & O & X_{t+1} \end{bmatrix}, \quad \tilde{\epsilon}_t = \begin{bmatrix} \epsilon_{t-1} \\ \epsilon_t \\ \epsilon_{t+1} \end{bmatrix}.
\]

Then, we have

\[
\tilde{y}_t = \tilde{X}_t \tilde{\epsilon}_t + \tilde{\eta}_t
\]

with

\[
\text{Cov} \ \tilde{\epsilon}_t \tilde{\epsilon}_t = \tilde{\eta}_t \tilde{\eta}_t^T = \begin{bmatrix} I_{G_{t-1}} & O & O \\ O & v_t I_{G_t} & O \\ O & O & v_t I_{G_{t+1}} \end{bmatrix} \tilde{\eta}_t \tilde{\eta}_t^T.
\]

\[
D_t = \begin{bmatrix} I_{G_{t-1}} & O & O \\ O & v_t I_{G_t} & O \\ O & O & v_t I_{G_{t+1}} \end{bmatrix}, \quad \tilde{\eta}_t = \begin{bmatrix} \tilde{\eta}_{t-1} \\ \tilde{\eta}_t \\ \tilde{\eta}_{t+1} \end{bmatrix},
\]

where

\[
v_t = \begin{bmatrix} \epsilon_t \\ \epsilon_t \end{bmatrix}, \quad \tilde{v}_t = \begin{bmatrix} \tilde{\epsilon}_t \\ \tilde{\epsilon}_t \end{bmatrix}.
\]

For the moment, write

\[
\tilde{y}_t = \tilde{X}_t \tilde{\epsilon}_t + \tilde{\eta}_t
\]
Then, if the error term $\beta_t$ is normally distributed, the log likelihood function $\log L$ can be written as

$$\log L = \text{const} - \frac{1}{2} \log |\bar{\beta}| - \sum_{k} G_{t,1k} \log v_{t,k} - \frac{1}{2} \log |\bar{\beta}| - \frac{1}{2} \sum_{k} \left( \bar{\beta}_t - \bar{\beta}_t \right) \bar{\beta}_t$$

In this case, the maximum likelihood estimators of $\bar{\beta}_t$ and $\bar{\beta}_t$ are, as usual, given by

$$\hat{\beta}_t = \frac{\bar{\beta}_t}{G_{t,1t} \bar{\beta}}$$

By substituting $\bar{\beta}_t \beta$ to $\bar{\beta}_t \beta$, we see that the concentrated log maximum likelihood function is, except for a constant, equal to

$$\log L = \text{const} - \frac{1}{2} \log |\bar{\beta}_t| - \sum_{k} G_{t,1k} \log v_{t,k} - \frac{1}{2} \log |\bar{\beta}_t| - \frac{1}{2} \sum_{k} \bar{\beta}_t \bar{\beta}_t$$

where $\bar{\beta}_t = \frac{\bar{\beta}_t}{G_{t,1t} \bar{\beta}}$. The maximum likelihood estimators of $\bar{\beta}_t$ and $\bar{\beta}_t$ are then defined to be the maximizers of $\log L$. However, the size of the matrix $\bar{\beta}_t$ is typically large and we cannot expect to compute numerically its determinant with satisfactory accuracy. On the other hand, we see that $\bar{\beta}_t$ depends only on $\bar{\beta}_t$ and is bounded. Therefore, ignoring the first term of $\log L$, we propose to estimate $\bar{\beta}_t$ and $\bar{\beta}_t$ by maximizing

$$\log L = \text{const} - \sum_{k} G_{t,1k} \log v_{t,k} - \frac{1}{2} \bar{\beta}_t \bar{\beta}_t$$

Our method is analogous to the quasi maximum likelihood estimation of time series model, and, as is usual, it is expected to be efficient for a larger class of distribution than the normal distributions. Once we obtain estimates of $\bar{\beta}_t$, $v_{t}$, and $v_{t}$, we may estimate $\bar{\beta}_t$ and $\bar{\beta}_t$ by using $\bar{\beta}_t \beta$

### 4 The Data and Empirical Result

In this research, we have used the monthly data of the closing prices of the Japanese government fixed-coupon bonds from September to August. We exclude the bonds with maturity shorter than one year and longer than twenty years since the former are susceptible to the effect of the monetary policy while the latter have low trading volume. In our data set,
there are bonds on average in each month with coupon rate ranging from to per annum. We estimated the model by using the method described in Section . For instance, by using the bond price data of September, October and November of , we obtain the parameter estimates for October .

Figure shows the cross-sectional performance of our model. The first figure shows the fluctuation of the estimated , while RSD in the second figure denotes the residual standard deviation of the model given by

\[
\text{RSD}_{t} = \sqrt{\frac{1}{G_t} \sum (y_t - \hat{y}_t, \hat{y}_t - X_t \beta_t, \mu_t).}
\]

From these figures, we see that except for a couple of years around the Lehman shock in , our model performed well with RSD less than . The maximum RSD is in November . The estimated shows a similar fluctuation to RSD. The overall performance of the model is similar to that of . This supports the importance of the cross-sectional correlation structure of the stochastic discount functions. Notice, in particular, that in , decreased whereas RSD slowly but increased. This implies that in this period, the overall magnitude of the stochastic discount functions was small but the correlation among them was large.

Figure shows the time-series fluctuation of the polynomial coefficients of the mean discount functions . From this figure, we can say that these coefficients are highly correlated to each other. It is observed that the coefficients of odd orders are positively correlated to those of odd orders, and negatively correlated to those of even odd, respectively, which prevents the slope of the term structure of the dis-
count functions from being extremely large or low. We also see that in the long run, there are possibility of some trend or non-stationarity in these coefficients but in a shorter period such as one year, we may regard the coefficients stationary.

Figure 1 and 2 show the time-series fluctuation of the cross-sectional and time-series covariance structures. From these figures, we see that the discount functions are positively correlated both in view of cross-section and time-series. In addition, it can be seen that during the financial crisis period of length about two years around the Lehman shock, both cross-sectional and time-series correlations among the bonds are high. Recall that in this period, $M_t$ and $RSD$
were both high as well. Therefore, we may conclude that the magnitude of uncertainty in the term structure of interest rate consisted of a small number of, but highly volatile, random factors in this financial crisis period. On the other hand, as we have indicated earlier by comparing the fluctuations of $\Delta_i$ and RSD, the cross-sectional and time-series correlation are high in $\bar{u}$ as well. Notice however that both $\Delta_i$ and RSD were relatively small in $\bar{u}$. We may conclude therefore that although European financial crisis were worried in this period, the nature of its influence on the price movement of the Japanese government bonds was somewhat different from that of the subprime loan-related problems and the Lehman shock. In conclusion, we can say that both cross-sectional and time-series covariance structure in the discount functions provide important information for analysis of the bond price movements.

5 One-Month Forecast of Bond Prices

In this section, we present a bond price forecasting method using our model as well as the forecast performance of the Japanese government bonds. As we have described, in our model, the time-series bond price fluctuations are captured by the time varying regression coefficients and by the time-series covariance structure of the error term of the regression. We can then forecast bond prices based on these two components.

Suppose that we are interested in forecasting the bond price at time $T'$ based on the observations from time period $t = \{T' - m\}$. Then, we can estimate the parameters $\bar{\Omega}_i$, $\bar{\Delta}_i$ and $\bar{\tau}_i$ for $t = \{T' - m\}$ by the estimation method described in Section \textcopyright. Notice that we can also
estimate $\hat{\theta}_T$, $\tilde{\theta}_T$ and $\tilde{v}_T$ based on the observations at time $T$ - $\Box$ and $T$ only, in the same manner as in Section \textsection 8. Let $\hat{\theta}_t$, $\tilde{\theta}_t$, and $\tilde{v}_t$, $t = \Box$ be the estimated parameters obtained in such a way. First, we forecast $\hat{\theta}_{T+1}$ by fitting some time-series model to the estimated $\hat{\theta}_t$, $\tilde{\theta}_t = \Box$ and $\tilde{v}_t$, $t = \Box$. Although we may adopt any kind of time-series models for this purpose, we have to consider the maximum number of the parameters of the time-series model, which in turn is restricted by the number $T$ of observations. In practice, it is reasonable not to take too long a period because of the possibility of structural changes. Taking such a restriction on the sample period in mind, we propose to extract a small number of principal components from the observed $\hat{\theta}_t$, and fit a univariate time-series model of a low order such as AR $\Box$ to each of the extracted principal components. In other words, let $\hat{\Theta}$ be the sample covariance matrix of the estimated beta, that is

$$
\hat{\Theta} = \frac{1}{T} \sum_{t=1}^{T} \hat{\theta}_t \tilde{\theta}_t' - \hat{\Theta}_1 - \hat{\Theta}_2
$$

and let $\hat{\lambda}_1 > \Box > \Box$ and $\hat{\Phi}_1, \hat{\Phi}_2, \ldots$, be the eigenvalues and corresponding normalized eigenvectors of $\hat{\Theta}$. The $i$-th principal component $z_{it}$ of the estimated $\hat{\Theta}$ is then given by

$$
z_{it} = \sum_{j=1}^{p} \hat{\Phi}_{ij} \hat{\lambda}_{ij}.
$$

Let $k$ be the number of the principal components for which the cumulative rate of contribution first exceeds some preassigned level, say $\Box$, that is

$$
k = \min \left\{ m \leq p \mid \frac{\sum_{i=1}^{k} \hat{\lambda}_{ij}}{\sum_{i=1}^{p} \hat{\lambda}_{ij}} > \Box \right\}.
$$

Then, we fit some univariate time-series model to each of $z_{it}$, $i = \Box$, $t = \Box$. For instance, we may fit an AR $\Box$ to each of $z_{it}$ as

$$
z_{it} = \hat{\lambda} z_{it-1} + \tilde{\lambda} i, t = \Box. \Box.
$$

In this case, let

$$
\hat{\lambda}, \tilde{\lambda}, \hat{\lambda}, \tilde{\lambda} t = \Box, \Box, \Box, \Box, \Box,
$$

be the estimates of $\hat{\lambda}$, so that we may forecast $z_{ir+1}$ by

$$
z_{iT+1} = \hat{\lambda} z_{iT}.
$$

We may then forecast $\hat{\theta}_{T+1}$ by

$$
\hat{\theta}_{T+1} = \sum_{j=1}^{k} \sqrt{\hat{\lambda}_j} \tilde{\lambda}_j \hat{\lambda}_j \tilde{\lambda}_j T + \Box.
$$

Next, let $\hat{\theta}_T$ be the residual of the regression $\Box$ at time $t = T$, that is $\Box$.
In order to forecast the bond price at time \( T \), we have to forecast \( \Pi_T \). Using the estimated parameters \( \hat{\alpha}_T \) and \( \hat{\phi}_T \), we estimate the covariance of \( \Pi_T \) and \( \Pi_{T+1} \) by \( \hat{\Pi}_{TT} \), with \( \hat{\Pi}_{TT} \) computed by substitution of estimated \( \hat{\alpha}_T \) into \( \Pi_T \). Then, we may forecast \( \Pi_{T+1} \) by

\[
\Pi_{T+1} = \hat{\Pi}_{TT} \hat{\alpha}_T + \hat{\phi}_T \Pi_T.
\]

We finally forecast the bond price at time \( T \) by

\[
\hat{P}_T = \hat{\alpha}_T + \hat{\phi}_T \hat{X}_T \hat{\Pi}_T \hat{\alpha}_T + \hat{\phi}_T \Pi_T.
\]

By using the forecast method described in the above, we have investigated the forecast performance of the Japanese government bond prices. For each month \( T \) from August \( T \) to August \( T \), we have computed bond price forecast at \( T \) based on the observations for \( t = T - T \). The forecast performance has been measured by the mean squared error \( \text{MSE} \), which is defined by

\[
\text{MSE}_T = \frac{1}{G_T} \sum_{g} \left( P_{gT} - \hat{P}_{gT} \right)^2.
\]

Here, \( P_{gT} \) denotes the observed bond price and \( \hat{P}_{gT} \) denotes the forecasted price of the bond. Figure \( \hat{g} \) shows the MSE of our model for the Japanese government bonds. It can be seen that the MSE is less than \( 0 \) Japanese yen \( T \) most of the time. The performance is not good when the bond price fluctuation is high, such as in December \( \hat{g} \). Figure \( \hat{g} \) also shows Figure \( \hat{g} \) as well. It should be noted, however, that even in such a case, the forecast performance tends to improve rapidly within a few months.

Figures \( \hat{g} \) and \( \hat{g} \) show the forecast performance for bond portfolios with different terms. For the convenience, we have construct four bond portfolios of short, medium, long and super-long terms. The components of these bond portfolios are given by Table \( \hat{g} \). For instance, the portfolio of short term bonds consists of the bonds with time to maturity from one to three years. From these figures, we see that the forecasting performance is good enough for the short and medium term bonds. It is relatively difficult to forecast the prices of the long and super-long term bonds since the price movement of these bonds are relatively large. In particular, it seems that the bonds with super long term, which are less liquid than other bonds, tend to be
Figure Mean-Squared Error of The Bond Forecast

Figure The Forecast Performance of the Bond Portfolios Short and Medium Terms

Figure The Forecast Performance of the Bond Portfolios Long and Super Long Terms
preferred when the financial market is instable, which makes the price of these bonds rise too rapidly to follow.

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