Dynamic characterization of coupled nonlinear oscillators caused by the instability of ionization waves

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We have experimentally investigated the dynamic behavior of coupled nonlinear oscillators, including chaos caused by the instability of ionization waves in a glow discharge plasma. We studied the phase synchronization process of coupled asymmetric oscillators with increasing coupling strength. Coherence resonance and phase synchronization were observed in the coupled systems. The phase synchronization process revealed scaling laws with a tendency of Type-I intermittency in the relationships between the coupling strength and the average duration of successive laminar states interrupted by a phase slip. Coupled periodic oscillators changed from a periodic state to chaos caused by the interaction of nonlinear periodic waves at increasing coupling strength.

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I. INTRODUCTION

Over the past few decades, coupled nonlinear oscillators have attracted much attention and have been investigated in a wide range of fields. Coupled nonlinear oscillators have the potential for widespread applications. For coupled nonlinear oscillators, a variety of phenomena such as chaos synchronization and coherence resonance have been observed. Many nonlinear and non-equilibrium phenomena exist in a plasma; therefore, a plasma is an optimal medium for testing the universality of nonlinear and non-equilibrium phenomena.

The study of coupled nonlinear oscillators is also of consequence in plasma physics. In Popov’s study, a theoretical and numerical investigation concerning the synchronization in coupled beam-plasma systems was presented. In Ref. 13, the spatiotemporal synchronization of chaos in coupled symmetric oscillators caused by the instability of ionization waves in a glow discharge plasma was experimentally observed. However, further detailed studies on the dynamic behavior of coupled nonlinear oscillators are required. In Ref. 14, the interaction between highly coherent periodic oscillations in a glow discharge plasma was studied, and interesting results such as frequency synchronization and frequency pulling on the nonlinear dynamics were reported. In this study, we focus on chaotic behavior and experimentally investigate the properties of coupled nonlinear oscillators, including chaos caused by the instability of ionization waves in a glow discharge plasma.

The paper is organized as follows: The experimental configuration is described in Sec. II, whereas the results concerning the process of phase synchronization (PS) and coherence resonance (CR) of two coupled chaotic oscillators according to increasing coupling strength are reported in Sec. III. Results concerning the process that causes the chaotic behavior of two coupled nonlinear oscillators with periodic orbits according to increasing coupling strength are given in Sec. IV. Finally, the findings are summarized in Sec. V.

II. EXPERIMENTAL CONFIGURATION

A schematic representation of the experimental setup used to study nonlinear wave-wave interactions in a laboratory plasma is shown in Fig. 1. A series of experiments were performed using two glass tubes with the diameter and length of 0.02 m and 0.75 m, respectively. Neon gas was introduced into each tube at a pressure of approximately 478 Pa, after evacuating each tube to a high vacuum. When a high dc potential was applied to the electrodes, a Ne plasma was produced as a glow discharge between the electrodes. Resistors with a resistance of 4.0 kΩ were installed in each circuit in order to sustain the discharge. Ionization waves then propagate from the cathode to the anode. The ionization waves are self-excited and unstable owing to plasma instability. Because the direction of the phase velocity is opposite to that of the group velocity, the ionization waves excited in these experiments have backward wave properties.

FIG. 1. Schematic representation of the experimental setup. The pressure in each tube is approximately 478 Pa, and the discharge currents in tubes 1 and 2 are changed. Resistors with a resistance of about 4.0 kΩ were installed in each circuit in order to sustain discharge. The two tubes were electrically connected using a variable resistor. Waves 1 and 2 were sampled from tubes 1 and 2, respectively. The time series signals were obtained as fluctuations in the light intensity using photodiodes. The photodiodes on tubes 1 and 2 were placed 0.184 m from the anode.
The typical electron and ion temperatures in the plasmas were approximately 10 eV and 0.025 eV, respectively. The discharge currents in tubes 1 and 2 were varied to govern the environments of the systems. Waves 1 and 2 were sampled from tubes 1 and 2, respectively. The two nonlinear oscillators, i.e., plasma tubes, were operated independently. The two unstable waves interacted with each other through coupling by a variable resistor. The value of the resistor connecting the two oscillators is a control parameter that governs the coupling strength of the nonlinear oscillators. The time series signals for the analysis were obtained as fluctuations in the light intensity using photodiodes (S6775, HAMAMATSU) and were sampled using a digital oscilloscope (GDS-1072A-U, GWINSTEK). The photodiodes on tubes 1 and 2 were placed 0.184 m from the anode. When the values described above were selected as the discharge current and gas pressure (in this experiment, the gas pressure is fixed at 478 Pa), the system exhibited a wide variety of oscillations, including periodic and chaotic oscillations. Here, $I$ stands for the value of the discharge current.

Table I shows the dynamics versus the discharge current in tubes 1 and 2. It should be noted that tubes were operating under slightly different conditions, for example, the discharge source devices were not identical, and therefore, there is machine dependence. For these reasons, the dynamic behavior of the discharge current differs between the two tubes.

### III. PHASE SYNCHRONIZATION AND COHERENCE RESONANCE IN COUPLED CHAOTIC OSCILLATORS

As mentioned in the introduction, chaotic oscillators can synchronize through an interaction. When two chaotic oscillators interact with each other, the coupled systems can show various behaviors such as phase synchronization, lag synchronization, and complete synchronization. Studies on coupled symmetric oscillators are performed as mentioned above. On the other hand, the study of coupled asymmetric oscillators has accomplished little. It often occurs in spatially extended systems such as plasmas and chemical reactions in nature that asymmetric oscillators interact with each other. Therefore, the study of coupled oscillators with an asymmetry in each system is essential and has the possibility of discovering related phenomena.

When the discharge currents in tubes 1 and 2 are fixed at 28.0 mA and 8.0 mA, respectively, each wave shows chaotic oscillations as shown in Fig. 2. The two tubes exhibit asymmetric current values. The values of discharge current in the two tubes are steady-state and settled as the coupled oscillators become extremely asymmetric. Figure 2 shows (a) the time series signals, (b) the power spectrum, (c) the trajectories reconstructed in phase space using the embedding method advocated in Ref. 19, and (d) the phase difference $|\Phi_1 - \Phi_2|$ between waves 1 and 2 with respect to the time before coupling. Phase differences $|\Phi_1 - \Phi_2|$ between two oscillators are calculated as follows: The period is defined as the time from a certain maximum value to the subsequent maximum, i.e., peak-to-peak in time series. Only maximum value points in waves 1 and 2 are extracted. The times corresponding to $\rho_{th}$ maximum values of wave 1 and 2 are compared, and then the phase difference $|\Phi_1 - \Phi_2|$ is calculated in sequence from the results. The time series shows a chaotic oscillation, the power spectra have disorder peaks, and no relation is found between phases of waves 1 and 2 because the value of $|\Phi_1 - \Phi_2|$ increases with time. This value increases gradually; therefore, the two oscillators do not synchronize.

The two ionization waves interact through the resistor between the two oscillators, and then as the value of the resistor is decreased, the coupling strength increases and the coupled systems reach PS. Figure 3 shows the coupled systems in the process of coupling, which is the PS state, because the value of $|\Phi_1 - \Phi_2|$ does not increase as in the previous case and remains almost constant with respect to time. The value of the resistor, which is the control parameter for the coupling strength, is approximately 0 kΩ. It is shown that the frequency around 1.2 kHz and 2.4 kHz of wave 1 is especially emphasized and becomes dominant in the process of coupling as shown in the power spectrum of Fig. 3. The reconstructed trajectories shown in Fig. 3 form simple loops. Therefore, this analysis reveals that the coherence resonance and phase synchronization occur at the same time in asymmetric coupled oscillators.

To examine the change of dynamic behavior when the coupling strength is varied quantitatively, the Lyapunov exponents were calculated as a chaotic analysis. Here, the largest Lyapunov exponents were calculated using a time series sampled in the experiments, based on the algorithm advocated in Ref. 20. The largest Lyapunov exponents were calculated from the time series sampled from the photodiodes on tubes 1 and 2. For chaotic oscillations, the value of the largest Lyapunov exponent is positive; this value is higher for a more chaotic system. The value comes close to zero for a system with periodic oscillations. Figure 4 shows the relationships between the value of the resistor and the largest Lyapunov exponents sampled from waves 1 and 2. The largest Lyapunov exponent of wave 1 suddenly decreases around 2 kΩ and eventually becomes close to 0. This value, $\approx 2$ kΩ, is the threshold for the transition from a chaotic to a periodic state. On the other hand, the largest Lyapunov exponents of wave 2 maintain a certain value.
For these reasons, this analysis shows that the shape of wave 1 suddenly becomes ordered below 2 kΩ, and the tube 1 system (wave 1) is more easily influenced than that of tube 2 (wave 2). Wave 1 in the coupled systems shows periodic oscillations that exhibit highly coherent behavior with increasing coupling strength, i.e., the system shows coherence resonance. We observe that the periodic oscillation is created because of the interaction of the chaotic oscillation. This phenomena means that coupled nonlinear oscillators can possibly be used to control chaos.

Figure 5 shows the relationships between the value of the resistor and the slope of the phase difference between waves 1 and 2, \( \frac{\Delta \Phi_1 - \Phi_2}{N} \). The slope of the phase difference was calculated by the least squares method. From Fig. 5, the slope repeatedly approaches zero with the decreasing resistor value, i.e., PS occurs repeatedly. Here, it should be noted that PS occurs between 3 and 5 kΩ, but the system does not show periodic oscillation, i.e., the coherent resonance does not occur. This is also clear from the calculated result of the largest Lyapunov exponent as shown in Fig. 4, i.e., the largest Lyapunov exponent comes close to zero below 2 kΩ.

Figure 6(a) shows an enlarged view of the phase difference. Here, the data of the phase difference \( \frac{\Delta \Phi_1 - \Phi_2}{N} \) between waves 1 and 2 were interpolated based on the method described in Ref. 23. On the threshold of PS, the synchronized (laminar) state is interrupted by a phase slip as shown in Fig. 6(a) and then continues intermittently. In coupled nonlinear oscillators, the transition to PS can be considered a saddle-node bifurcation. The relationship between the coupling strength \( R \) and the average duration \( \langle \ell \rangle \) of successive laminar
FIG. 3. (a) Time series signals, (b) power spectrum, (c) the trajectories reconstructed in phase space, and (d) the phase difference $|\Phi_1 - \Phi_2|$ between waves 1 and 2 with respect to time in the process of coupling. The value of the resistor is approximately 0 kΩ.

FIG. 4. Relationships between the value of the resistor and the largest Lyapunov exponents sampled from waves 1 and 2.

FIG. 5. Relationships between the value of the resistor and the slope of the phase difference between waves 1 and 2, $\frac{\Delta \Phi_1}{\Delta t}$. 
states interrupted by phase slips follows a Type-I intermittency scaling law: $^{23-25} (\ell) \sim |R - R_0|^{-\frac{1}{2}}$. Here, $R_0$ denotes the value for the transition to PS.

Figure 6(b) shows the extended version of the relationships between the value of the resistor and the slope of the phase differences. It should be noted that the shape of extended version is slightly different from that shown in Fig. 5 because the system is very sensitive to the experimental conditions. Here, we focus on the circled region where the system transits to PS for $3.6 \text{k}\Omega < R < 3.9 \text{k}\Omega$, and the value for the transition to PS is selected as $R_0 = 3.6 \text{k}\Omega$. The PS occurs when the slope of the phase difference decreases continuously. The slope of the phase difference repeatedly decreases with the decreasing resistor value as shown in Fig. 5. We focus on the regime of $R$ where $3.6 \text{k}\Omega < R < 3.9 \text{k}\Omega$, where the slope decreases gradually. Therefore, $R_0$, the value for the transition to PS, is selected as $3.6 \text{k}\Omega$.

Figure 6(c) shows the average duration $\langle \ell \rangle$ of successive laminar states interrupted by phase slips for $|R - 3.6|^{-\frac{1}{2}}$. From this result, we find that the scaling laws have a tendency for Type-I intermittency.

IV. APPEARANCE OF CHAOS CAUSED BY WAVE-WAVE INTERACTION OF NONLINEAR PERIODIC OSCILLATORS

The dynamic behavior of the coupled oscillators is investigated when the nonlinear periodic oscillators interact with each other. Under linear conditions, the interaction of the periodic oscillators never leads to chaos. However, with nonlinearity, the coupled oscillators are able to change from a periodic to a chaotic state caused by the interaction of the nonlinear periodic waves according to the changing coupling strength.

When the discharge currents in tubes 1 and 2 were fixed at 20.0 mA and 19.0 mA, respectively, each wave displayed periodic oscillation as shown in Fig. 7. The time series signals and power spectrum before coupling are shown in Fig. 7. The time series shows periodic oscillation, and the power spectra have sharp peaks. The two ionization waves interact through the resistor and then, with the decreasing value of the resistor, i.e., increasing coupling strength, the coupled systems become chaotic. Figure 8 shows the coupled systems in the process of coupling, i.e., the chaotic state. The value

![FIG. 6. Relationships between the value of the resistor and the average duration of the successive laminar state interrupted by phase slips are shown. (a) Enlarged view of the phase difference, (b) relationships between the value of the resistor and the slope of the phase differences (extended version), and (c) relationships between the value of the resistor and the average duration of successive laminar states interrupted by phase slips.](image)

![FIG. 7. Time series signals and power spectrum before coupling. The discharge currents in tubes 1 and 2 were 20.0 mA and 19.0 mA, respectively.](image)
of the resistor, which is the control parameter of the coupling strength, was approximately 0 kΩ.

The largest Lyapunov exponents were calculated from the time series sampled from the photodiodes on tubes 1 and 2. Figure 9 shows the relationships between the value of the resistor and the largest Lyapunov exponents sampled from waves 1 and 2. The largest Lyapunov exponent of wave 2 immediately increases, i.e., wave 2 becomes chaotic immediately after the two ionization waves are coupled. The largest Lyapunov exponent of wave 1 suddenly increases around 2.0 kΩ and eventually closely approaches the value of wave 2. This value, around 2.0 kΩ, is the threshold of transition from a periodic to a chaotic state.

Figure 10 shows the relationships between the value of the resistor and the slope of the phase difference between waves 1 and 2, \( \frac{\Delta \phi_1 - \Delta \phi_2}{\Delta t} \), when the coupling strength increases, i.e., the value of the resistor decreases. From Fig. 10, we see that the slope immediately increases below 2.0 kΩ. When these phenomena are considered together with the largest Lyapunov exponent as shown in Fig. 9, it is clear that the coupled system roughly shows PS above 2.0 kΩ and that the largest Lyapunov exponent increases below 2.0 kΩ, i.e., both oscillators become chaotic and the phase between the waves desynchronizes. It is understood that the chaotic behavior is caused by the collapse of a T2 torus consisting of two frequencies. This is the mechanism advocated by Curry and Yorke.26

V. CONCLUSION

In this study, we have experimentally investigated the dynamic characterization of coupled oscillators caused by the instability of ionization waves in a glow discharge plasma. When the dynamic behavior of the two asymmetric chaotic oscillators in the process of coupling was studied according to increasing coupling strength, PS and coherence resonance were observed in the coupled system. In the process toward PS of the coupled chaotic oscillators, we found scaling laws that had a tendency to Type-I intermittency of the relationships between the coupling strength and the average duration of successive laminar states interrupted by
phase slips. When we studied the dynamic behavior of the two nonlinear oscillators with periodic orbits in the process of coupling as a function of increasing coupling strength, we found that the coupled oscillators change from a periodic to a chaotic state.