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<td>Citation</td>
<td>長崎大学工学部研究報告 Vol.30(54) p.29-36, 2000</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2000-01</td>
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<tr>
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Plane Coutte-Poiseuille Flow of Power-Law Non-Newtonian Fluids

by
Ganbat DAVAA*, Toru SHIGECHI** and Satoru MOMOKI**

The fully developed laminar flow of a non-Newtonian fluid flowing between two parallel plates with one moving plate was studied analytically. Applying the shear stress described by the power-law model, the exact solutions for the momentum equation were obtained.

The effects of the velocity of a moving plate and the flow index of a non-Newtonian power-law fluid on the velocity distribution and friction factor have been discussed.

1. Introduction

Problems involving fluid flow and heat transfer with an axially moving core of solid body or fluid in an annular geometry can be found in many manufacturing processes, such as extrusion, drawing and hot rolling, etc. In such processes, a hot plate or cylindrical rod continuously exchanges heat with the surrounding environment. For such cases, the fluid involved may be Newtonian or non-Newtonian and the flow situations encountered can be either laminar or turbulent.

In the previous report[1], fully developed laminar heat transfer of a Newtonian fluid flowing between two parallel plates with one moving plate was analyzed taking into account the viscous dissipation of the flowing fluid.

In engineering applications such as manufacturing processes, many important fluids are non-Newtonian in their flow characteristics.

In this paper, an exact solution of the momentum equation is obtained for fully developed laminar flow of a non-Newtonian fluid flowing between two parallel plates with one moving plate. The constitutive equation (i.e., the shear stress - shear rate relation) for a non-Newtonian fluid is described by the power-law model most frequently used in non-Newtonian fluid flow and heat transfer. The effects of the relative velocity of a moving plate and the flow index of a power-law fluid on the velocity distribution and friction factor have been discussed.

Nomenclature

- \( C \) integration constant
- \( p \) pressure
- \( u \) axial velocity of fluid
- \( u^* \) dimensionless velocity = \( u/u_m \)
- \( u_m \) average velocity of fluid
- \( U \) axial velocity of the moving plate
- \( U^* \) relative velocity of the moving plate
- \( y \) coordinate normal to the fixed plate
- \( y^* \) dimensionless coordinate = \( y/L \)
- \( z \) axial coordinate
- \( \rho \) density
- \( L \) channel width
- \( m \) consistency index
- \( n \) flow index
- \( f \) friction factor
- \( F \) dimensionless parameter
- \( Re^* \) generalized Reynolds number

Subscripts

- 0 fixed plate
- L moving plate

2. Analysis

The physical model for the analysis is shown in Fig.1. The assumptions and conditions used

Received on October 26, 1999

* Graduate Student, Department of Mechanical Systems Engineering
**Department of Mechanical Systems Engineering
in the analysis are:

1. The flow is incompressible and steady-laminar, and hydrodynamically fully developed.
2. The fluid is non-Newtonian and the shear stress may be described by the power-law model, and physical properties are constant.
3. Either of two parallel plates is axially moving at a constant velocity.

The governing momentum equation together with the assumptions described above is

$$\frac{d\tau}{dy} = -\frac{dp}{dz}$$  \hspace{1cm} (1)

The boundary conditions are:

$$\begin{cases} 
 u = 0 & \text{at } y = 0 \\
 u = U & \text{at } y = L 
\end{cases}$$  \hspace{1cm} (2)

The shear stress on the left hand side of Eq.(1), \(\tau\), is given by the power-law model.

$$\tau = -m \left| \frac{du}{dy} \right|^{n-1} \frac{du}{dy}$$  \hspace{1cm} (3)

For a Newtonian fluid, \(n = 1\) and \(m\) coincides with the ordinary viscosity.

The friction factor, \(f\), and generalized Reynolds number, \(Re^*\), are defined as

$$f \equiv \frac{L}{\mu u_m} \left( -\frac{dp}{dz} \right)$$  \hspace{1cm} (4)

$$Re^* \equiv \frac{\mu u_m^{n-1}(2L)^n}{m}$$  \hspace{1cm} (5)

The average fluid velocity, \(u_m\), is defined as

$$u_m = \frac{1}{L} \int_0^L u dy$$  \hspace{1cm} (6)

The following dimensionless parameters are introduced:

$$y^* \equiv \frac{y}{L}$$  \hspace{1cm} (7)

$$u^* \equiv \frac{u}{u_m}$$  \hspace{1cm} (8)

$$U^* \equiv \frac{U}{u_m}$$  \hspace{1cm} (9)

$$L_{max}^* \equiv \frac{L_{max}}{L}$$  \hspace{1cm} (10)

For the case with a moving plate, two kinds of velocity profiles across the parallel plates' passage may be assumed as illustrated in Figs.1 and 2. The velocity profile shown in Fig.1 has a maximum at \(y = L_{max}\) whereas it has no maximum point in Fig.2. The two cases are respectively referred to as Case I and Case II.

Case I: The shear stress is calculated as

$$\tau = -m \left( \frac{du_a}{dy} \right)^n (0 \leq y \leq L_{max})$$  \hspace{1cm} (11)

$$\tau = m \left( -\frac{du_b}{dy} \right)^n (L_{max} \leq y \leq L)$$  \hspace{1cm} (12)

(a) \(0 \leq y \leq L_{max}\) \hspace{1cm} \(0 \leq y^* \leq L_{max}^*\)

The momentum equation and its boundary conditions are reduced to

$$\frac{d}{dy^*} \left( \frac{du^*_a}{dy^*} \right)^n = -F$$  \hspace{1cm} (13)

$$u^*_a = 0 \text{ at } y^* = 0$$  \hspace{1cm} (14)

where \(F\) is a parameter defined as

$$\frac{f \cdot Re^*}{2} = \frac{L_{max}^{n+1}}{m \cdot u_m^*} \left( -\frac{dp}{dz} \right) \equiv F$$  \hspace{1cm} (15)

Since the velocity gradient is zero at the location of maximum velocity,

$$\frac{du^*_a}{dy^*} = 0 \text{ at } y^* = L_{max}^*$$  \hspace{1cm} (16)

The integration of Eq.(13) together with Eq.(16) gives

$$\frac{du^*_a}{dy^*} = F \left( \frac{L_{max}^* - y^*}{L_{max}^*} \right)^{\frac{1}{n}}$$  \hspace{1cm} (17)

Integrating Eq.(17) together with Eq.(14), we have

$$u^*_a = \frac{n}{1+n} F \left[ L_{max}^{\frac{n+1}{n}} - (L_{max}^* - y^*)^{\frac{n+1}{n}} \right]$$  \hspace{1cm} (18)

(b) \(L_{max} \leq y \leq L\) \hspace{1cm} \(L_{max}^* \leq y^* \leq 1\)

The momentum equation and its boundary conditions are reduced to

$$\frac{d}{dy^*} \left( -\frac{du^*_b}{dy^*} \right)^n = F$$  \hspace{1cm} (19)

$$u^*_b = U^* \text{ at } y^* = 1$$  \hspace{1cm} (20)

Since the velocity gradient is zero at the location of maximum velocity,

$$\frac{du^*_b}{dy^*} = 0 \text{ at } y^* = L_{max}^*$$  \hspace{1cm} (21)

![Fig.1 Schematic of parallel plates with one moving plate for the case of velocity profiles assumed in this analysis (Case I)]
The integration of Eq.(19) together with Eq.(21) gives
\[ \frac{du^*}{dy^*} = -F^\frac{1}{1+n} (y^* - L_{\text{max}}^*)^\frac{1}{n} \] (22)
Integrating Eq.(22) together with Eq.(20), we have
\[ u^* = U^* + \frac{n}{1+n} F^\frac{1}{n} \left( (1-L_{\text{max}}^*)^\frac{1}{n} - (y^* - L_{\text{max}}^*)^\frac{1}{n} \right) \] (23)
The values of \( F \) and \( L_{\text{max}}^* \) remain unknown. They are determined below.

From the continuity of velocities at the location of maximum velocity:
\[ u^*_a = u^*_b \text{ at } y^* = L_{\text{max}}^* \] (24)
we have the first relationship between \( F \) and \( L_{\text{max}}^* \):
\[ F = \left[ \frac{U^*}{n+1} \left\{ \frac{1}{L_{\text{max}}^*} - \left( 1 - L_{\text{max}}^* \right)^\frac{1}{n} \right\} \right]^n \] (25)
From the mass balance between two plates:
\[ \int_0^1 u^* dy^* = \int_0^{L_{\text{max}}^*} u^*_a dy^* + \int_{L_{\text{max}}^*}^1 u^*_b dy^* = 1 \] (26)
we have the second relationship between \( F \) and \( L_{\text{max}}^* \):
\[ F = \left[ \frac{1 - (1-L_{\text{max}}^*) U^*}{n+1} \left\{ \frac{1}{L_{\text{max}}^*} + \left( 1 - L_{\text{max}}^* \right)^\frac{1}{n} \right\} \right]^n \] (27)
Combining Eq.(25) and Eq.(27), we have the following relationship among \( U^* \), \( L_{\text{max}}^* \) and \( n \).
\[ U^* = \left[ \frac{L_{\text{max}}^*}{1+2n} \left\{ \frac{1}{L_{\text{max}}^*} + \left( 1 - L_{\text{max}}^* \right)^\frac{1}{n} \right\} \right] \left( 1 - (1-L_{\text{max}}^*) \frac{1}{n} \right) \] (28)

**Case II:** The shear stress is calculated as
\[ \tau = -m \left( \frac{du^*}{dy^*} \right) \quad (0 \leq y \leq L) \] (29)
The momentum equation and its boundary conditions are reduced to
\[ \frac{d}{dy^*} \left( \frac{du^*}{dy^*} \right)^k = -F \] (30)
\[ \begin{aligned}
& u^* = 0 \text{ at } y^* = 0 \\
& u^* = U^* \text{ at } y^* = 1
\end{aligned} \] (31)
Integrating Eq.(30), we have
\[ \frac{du^*}{dy^*} = \frac{1}{F^\frac{1}{1+n}} (C - y^*)^\frac{1}{n} \] (32)
The integration of Eq.(32) gives
\[ u^* = \frac{n}{1+n} F^\frac{1}{n} \left[ C_{\text{s}}^\frac{1}{n} - (C - y^*)^\frac{1}{n} \right] \] (33)
where \( C \) is an integral constant. Applying the boundary conditions of Eq.(31) to Eq.(33), we have
\[ U^* = \frac{n}{1+n} F^\frac{1}{n} \left[ C_{\text{s}}^\frac{1}{n} - (C - 1)^\frac{1}{n} \right] \] (34)
From the mass balance between two plates:
\[ \int_0^1 \left[ \frac{n}{1+n} F^\frac{1}{n} \left\{ C_{\text{s}}^\frac{1}{n} + \left( 1 - y^* \right)^\frac{1}{n} \right\} \right] dy^* = 1 \] (35)
Integrating Eq.(35), we have
\[ \frac{n}{1+n} F^\frac{1}{n} \left[ C_{\text{s}}^\frac{1}{n} + \left( 1 - y^* \right)^\frac{1}{n} \right] dy^* = 1 \] (36)
thus, \( F \) is obtained as
\[ F = \left[ \frac{1}{1+n} \left( C_{\text{s}}^\frac{1}{n} + \frac{n}{1+2n} \left( (C-1)^\frac{1}{n} - C_{\text{s}}^\frac{1}{n} \right) \right) \right]^n \] (37)
Combining Eq.(34) and Eq.(37), we have the following relationship among \( C \), \( n \), and \( U^* \):
\[ U^* = \left[ \frac{C_{\text{s}}^\frac{1}{n} - (C-1)^\frac{1}{n}}{C_{\text{s}}^\frac{1}{n} + \frac{n}{1+2n} \left( (C-1)^\frac{1}{n} - C_{\text{s}}^\frac{1}{n} \right) \right] \] (38)
The numerical values of \( L_{\text{max}}^* \) for the Case I and Case II are calculated and given, respectively, in Tables 1 and 2. \( U^* \) is a critical value that shows the border between Case I and Case II, and given, from Eq.(28) with \( L_{\text{max}}^* = 1 \) as
\[ U^*_c = \frac{1+2n}{1+n} \] (39)
3. Results and Discussion

Figure 3 shows the effects of the relative velocity of the moving plate $U^*$ on the velocity profiles across the parallel plates for the cases of $n = 0.2, 0.5, 1.0$ and $1.5$. The case of $n = 1$ corresponds to that of a Newtonian fluid. It is seen clearly in the figures that the profiles of the fluid velocity are deformed by the moving plate with a relative velocity $U^*$. Figure 4 shows the effects of the flow index $n$ of the power-law fluid on the velocity profiles across the parallel plates for the cases of $U^* = -1.5, 0, 1.0$ and $1.5$. The case of $U^* = 0$ corresponds to that of both plates fixed. It is seen in the figures that for $U^* < 0$ the velocity profile is parabolic having a larger maximum value with increasing values of $n$. The velocity profiles are strongly affected by the flow index $n$. For $U^* > 0$ the velocity profiles become linear as the effect of the axial pressure gradient in the fluid diminishes and the fluid flow is governed only by the shear flow induced by the moving plate. In this case, the effect of $n$ is rather weak.

The predicted friction factors in terms of ,
Fig. 3 Velocity profile
Fig. 4 Velocity profile
The effect of $n$ is to increase the value of $fRe^*$ with an increase in $n$.

Figure 6 shows the effect of $n$ on $fRe^*$, normalized by the value of $fRe^*(U^* = 0)$ for the case of $U^* = 0$. For $U^* > 0$, the ratio $fRe^*/fRe^*(U^* = 0)$ is always less than unity. The effect of $U^*$ becomes stronger in the region of larger values of $n$.

Table 3  Numerical values of $fRe^*$

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$n$ ($n > 1.0$). The numerical values of $fRe^*$ are given in Table 3.

4. Conclusion
The plane Coutte-Poiseuille flow of power-law non-Newtonian fluid was analysed.

The present study showed that for equal conditions:
1. The velocity profiles are strongly affected by the flow index, $n$, for the case of $U^* < 0$. In this case, the velocity profile is parabolic having a larger maximum value with increasing values of $n$. For the case of $U^* > 0$, the effect of $n$ on the velocity is small and the profile becomes linear.
2. The friction factor in terms of $fRe^*$ decreases with increasing values of $U^*$.

Reference