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<tr>
<td>Citation</td>
<td>長崎大学工学部研究報告 Vol.32(58) p.115-123, 2002</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2002-01</td>
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<td>URL</td>
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An analysis of bending problem of laminated rectangular and triangular plates

by

Mei Huang*  Takeshi Sakiyama**
Hiroshi Matsuda**  Chihiro Morita**

An approximate method is proposed for analyzing the bending problem of laminated rectangular and triangular plates. An equivalent rectangular plate is used to analyze the bending problem of triangular plates. Based on the first-order shear-deformation theory, the partial differential equations have been established. By making use of numerical integration, the solutions for these equations can be obtained in discrete forms. Numerical results are obtained for some laminated rectangular and triangular plates. The efficiency and accuracy of the numerical solutions by the proposed method are investigated.

1. INTRODUCTION

Due to the unique advantages of the composite materials, such as high strength-to-weigh ratio, high stiffness-to-weigh ratio, high fatigue resistance and low density, laminated plates are widely used in aerospace and aircraft industries, marine structures, automotive components and building structures. The analyses of laminated rectangular plates have been reported by many researchers. Kan and Ito [1] analyzed antisymmetric angle-ply rectangular plates by using Fourier series. Based on classical laminated plate theory, some numerical results were obtained for the plates with at least a pair of opposite edges simply supported. It is well known that classical laminated plate theory is only suitable for thin laminated plates since it neglects transverse shear deformations. In order to include the effects of shear deformation, other theories have been used. Whitney and Pagano [2] used first-order shear-deformation theory to study the bending and vibration problems of heterogeneous anisotropic rectangular plates. Phan and Reddy [3] analyzed laminated plates using a higher-order shear-deformation theory. Finite element results were presented for cross-ply plates, angle-ply plates and other symmetric plates. Panano and Hatfield [4] developed three-dimensional theory to get the analytical solutions of laminated rectangular plates. But as so far, little study of the bending problem of laminated triangular plates can be found.

In this paper, an approximate method is developed for analyzing the bending problem of laminated rectangular and triangular plates. An triangular plate can be considered as one kind of rectangular plate with nonuniform thickness. By transforming the differential equations into integral equations and applying numerical integration, the solutions for the partial differential equations can be obtained in discrete form. Based on first-order shear-deformation theory, numerical results are obtained for laminated rectangular and triangular plates and their accuracy is investigated.

2. Fundamental differential equations of a laminated rectangular plate

The coordinate system for a laminated plate used in the present study is shown in Figure 1. The xyz coordinate system is assumed to have its origin at the corner of the middle plane of the plate. The surfaces of the plate are at $z = \pm h/2$ and $h$ is the thickness of the
plate. $\theta$ is the angle of fibre orientation with respect to the $x$ axis, $1$-, $2$- and $3$-directions are principal axes in the longitudinal, transverse and normal directions, respectively. In this research, it is assumed that the transverse deflection is small, so that the out-of-plane components of the in-plane results $N_x$, $N_y$ and $N_{xy}$ are negligible. The differential equations are as follows:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad \text{(1a)}$$
$$\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad \text{(1b)}$$
$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - q + \sum_{c=0}^{n} \sum_{d=0}^{m} P_{cd} \delta(x-x_c) \delta(y-y_d) = 0 \quad \text{(1c)}$$
$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} - Q_x + \sum_{c=0}^{n} \sum_{d=0}^{m} P_{cd} \delta(x-x_c) \delta(y-y_d) = 0 \quad \text{(1d)}$$
$$\frac{\partial M_x}{\partial y} + \frac{\partial M_y}{\partial x} - Q_y + \sum_{c=0}^{n} \sum_{d=0}^{m} P_{cd} \delta(x-x_c) \delta(y-y_d) = 0 \quad \text{(1e)}$$
$$M_x = B_{11} \frac{\partial u}{\partial x} + B_{12} \frac{\partial v}{\partial y} + B_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + D_{11} \frac{\partial \theta_x}{\partial x} + D_{12} \frac{\partial \theta_y}{\partial y} + D_{16} \left( \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right) \quad \text{(1f)}$$
$$M_y = B_{11} \frac{\partial u}{\partial x} + B_{12} \frac{\partial v}{\partial y} + B_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + D_{11} \frac{\partial \theta_x}{\partial x} + D_{12} \frac{\partial \theta_y}{\partial y} + D_{16} \left( \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right) \quad \text{(1g)}$$
$$M_{xy} = B_{11} \frac{\partial u}{\partial x} + B_{12} \frac{\partial v}{\partial y} + B_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + D_{11} \frac{\partial \theta_x}{\partial x} + D_{12} \frac{\partial \theta_y}{\partial y} + D_{16} \left( \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right) \quad \text{(1h)}$$
$$Q_x = k A_{14} \left( \frac{\partial w}{\partial y} + \theta_y \right) + k A_{15} \left( \frac{\partial w}{\partial x} + \theta_x \right) \quad \text{(1i)}$$

$\bar{Q}_{11} = \bar{Q}_{11} c^4 + 2 (\bar{Q}_{12} + 2 \bar{Q}_{66}) c^2 s^2 + \bar{Q}_{25} s^4$
$\bar{Q}_{12} = \bar{Q}_{12} (c^4 + s^4) + (\bar{Q}_{11} + \bar{Q}_{22} - 4 \bar{Q}_{66}) c^2 s^2$
$\bar{Q}_{16} = (\bar{Q}_{11} - \bar{Q}_{12} - 2 \bar{Q}_{66}) c^2 s^2 - (\bar{Q}_{22} - \bar{Q}_{12} - 2 \bar{Q}_{66}) c s^2$
$\bar{Q}_{22} = \bar{Q}_{12} s^4 + 2 (\bar{Q}_{12} + 2 \bar{Q}_{66}) c^2 s^2 + \bar{Q}_{26} c^4$
$\bar{Q}_{26} = (\bar{Q}_{11} + \bar{Q}_{22} - 2 \bar{Q}_{66}) c^4 s^2 + \bar{Q}_{66} (c^4 + s^4)$
$\bar{Q}_{44} = Q_{14} c^4 + Q_{55} s^4$
$\bar{Q}_{55} = Q_{15} c^4 + Q_{55} s^4$
$c = \cos \theta, \ s = \sin \theta$
$Q_{11} = \frac{E_1}{1 - v_{12} v_{21}}$
$Q_{12} = \frac{v_{12} E_2}{1 - v_{12} v_{21}}$
$Q_{22} = \frac{E_2}{1 - v_{12} v_{21}}$
$Q_{44} = G_{23}$
$Q_{66} = G_{31}, \ Q_{66} = G_{12}$

where $E_1$ is the axial modulus in the 1-direction, $E_2$ is
the axial modulus in the 2-direction, \( v_{12} \) is the Poisson's ratio associated with loading in the 1-direction and strain in the 2-direction, \( v_{12} \) is the Poisson's ratio associated with loading in the 2-direction and strain in the 1-direction, \( G_{23}, G_{31} \) and \( G_{12} \) are the shear moduli in 2-3, 3-1 and 1-2 planes.

The following non-dimensional expressions are used as:

\[
[X_1, X_3] = \frac{a^2}{D_b(1-v_{12}v_{21})}[Q_0, Q_3]
\]

\[
[X_2, X_0, X_1] = \frac{a}{D_b(1-v_{12}v_{21})}[M_0, M_0, M_1]
\]

\[
[X_6, X_5, X_3] = \left[ \begin{array}{c} \theta_0 \\ \theta_0 \\ \frac{w}{a} \end{array} \right]
\]

\[
[X_0, X_1] = \left[ \begin{array}{c} 0 \\ \frac{w}{a} \\ 0 \end{array} \right]
\]

\[
[X_{12}, X_{13}] = \frac{a^2}{D_b(1-v_{12}v_{21})}[N_0, N_0, N_2]
\]

\[
[\eta, \xi, \zeta] = \left[ \begin{array}{c} \frac{x}{a} \\ \frac{y}{b} \\ \frac{h}{h} \end{array} \right]
\]

where \( D_b = Eh_b^3/(12(1-v_{12}v_{21})) \) is the standard bending rigidity, \( h_b \) is the standard thickness of the plate.

By using the above expressions, the differential Eqs. (1a) - (1m) can be rewritten as

\[
\sum_{s=1}^{13} \left( \sum_{t=1}^{13} \sum_{i=1}^{m} \sum_{j=1}^{n} P_{tij} \delta(\eta_1 - \eta_1) \delta(\xi - \xi_2) \delta_{ij} = 0 \right)  
\]

where  \( t = 1 \sim 13 , \quad \bar{q} = \mu a^3 q / (D_b (1 - v_{12}v_{21})) \), \( \delta_{ij} \) is Kronecker's delta, \( F_{13}, F_{3t} \) and \( F_{3n} \) are given in Appendix A.

### 3 Discrete solutions of differential equations

By dividing a rectangular plate vertically into \( m \) equal-length parts and horizontally into \( n \) equal-length parts as shown in Figure 2, the plate can be considered as a group of discrete points which are the intersections of the \((m+1)\)-vertical and \((n+1)\)-horizontal dividing lines. In this paper, the rectangular area, \( \eta_1 \leq \eta \leq \eta_1, \ 0 \leq \xi \leq \xi_1 \) corresponding to the arbitrary intersection \((i, j)\) as shown in Figure 2 is denoted as the area \([i, j]\), the intersection \((i, j)\) denoted by \( \bullet \) is called the main point of the area \([i, j]\), the intersections denoted by \( \circ \) are called the boundary dependent points of the area.

By integrating the equation (2) over the area \([i, j]\), the following integral equation is obtained as

\[
\sum_{s=1}^{13} \left( \sum_{t=1}^{13} \sum_{i=1}^{m} \sum_{j=1}^{n} p_{tij} \delta(\eta_1 - \eta_1) \delta(\xi - \xi_2) \delta_{ij} = 0 \right)
\]

Next, by applying the numerical integration method, the simultaneous equation for the unknown quantities \( X_{i0} = X_i(\eta_i, \xi) \) at the main point \((i, j)\) of the area \([i, j]\) is obtained as follows

\[
\sum_{s=1}^{13} \left( \sum_{t=1}^{13} \sum_{i=1}^{m} \sum_{j=1}^{n} p_{tij} \delta(\eta_1 - \eta_1) \delta(\xi - \xi_2) \delta_{ij} = 0 \right)
\]

where \( p_{tij} = \alpha_{tk}/m, \quad \beta_{ij} = \alpha_{ij}/n, \quad \alpha_{tk} = 1 - (\delta_{tk} + \delta_{tk})/2, \quad \alpha_{ij} = 1 - (\delta_{ij} + \delta_{ij})/2, \quad t = 1 \sim 13, \quad i = 1 \sim m, \quad j = 1 \sim n, \quad \delta_{ij} \)
is the value of function $q(\gamma, \zeta)$ at point $(k, l)$.

The solution $X_{pij}$ of the simultaneous equation (4) is obtained as follows

$$X_{pij} = \sum_{t=1}^{13} \left[ \sum_{k=0}^{i} \beta_{ik} A_{pt} [X_{tk0} - X_{tki}(1 - \delta_{ik})] + \sum_{l=0}^{j} \beta_{jl} B_{pt} [X_{tli} - X_{tlj}(1 - \delta_{jl})] \right] + \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} C_{ptkl} X_{tkl}(1 - \delta_{ik} \delta_{jl})] + \sum_{k=0}^{i} \sum_{l=0}^{j} \beta_{ik} \beta_{jl} A_{ptl} q_{tkl} - \sum_{t=1}^{3} \sum_{c=0}^{m} \sum_{d=0}^{n} P_{tcd} u_{tc} u_{td}$$

where $p = 1 \sim 13$, $A_{pt}$, $B_{pt}$ and $C_{ptkl}$ are given in Appendix B.

In the equation (5), the quantity $X_{pij}$ at the main point $(i, j)$ of the area $[i, j]$ is related to the quantities $X_{tk0}$ and $X_{tli}$ at the boundary dependent points of the area and the quantities $X_{tkj}$, $X_{tji}$ and $X_{tkl}$ at the inner dependent points of the area. With the spreading of the area $[i, j]$ according to the regular order as $[1, 1]$, $[1, 2]$, $\ldots$, $[1, n]$, $[2, 1]$, $[2, 2]$, $\ldots$, $[2, n]$, $\ldots$, $[m, 1]$, $[m, 2]$, $\ldots$, $[m, n]$, a main point of a smaller area becomes one of the inner dependent points of the following larger areas. Whenever the quantity $X_{pij}$ at the main point $(i, j)$ is obtained by using the equation (5) in the above mentioned order, the quantities $X_{tkj}$, $X_{tji}$ and $X_{tkl}$ at the inner dependent points of the following larger areas can be eliminated by substituting the obtained results into the corresponding terms of the right side of equation (5).

By repeating this process, the equation $X_{pij}$ at the main point is only related to the quantities $X_{tk0}$, $X_{tli}$ and $X_{tkj}$ at the inner dependent points of the following larger areas can be eliminated by substituting the obtained results into the corresponding terms of the right side of equation (5).

The result is as follows

$$X_{pij} = \sum_{d=1}^{10} \left[ \sum_{t=0}^{i} a_{ptij} X_{tj0} + \sum_{l=0}^{j} b_{ptij} X_{tli} \right] + q_{pij} + \sum_{t=1}^{3} \sum_{c=0}^{m} \sum_{d=0}^{n} q_{tcdij} P_{tcd}$$

where $a_{ptij}$, $b_{ptij}$, $q_{pij}$ and $q_{tcdij}$ are given in Appendix C.

The equation (6) gives the discrete solution of the fundamental differential equation (2).

4 Equivalent rectangular plate of a triangular plate

An idea of equivalent rectangular plate is introduced to solve the bending problem of a laminated right triangular plates. A right triangular plate is quite different from uniform rectangular plates, but it can be translated into equivalent rectangular plates with non-uniform thickness (shown in Fig. 3). In order to ensure half part of the equivalent rectangular plate has the same boundary and load conditions as the original triangular plate, some point supports shown by ( ) at discrete points along the hypotenuse are used to enforce the boundary conditions along the original hypotenuse, the another half part of the equivalent rectangular plate with two free edges has very thin thickness and there is no load on it. The values of three reactions $P_{tij}$, $P_{tcd}$, $P_{tij}$ at each point support can be determined by the conditions $M_r = \theta_r = w = 0$.

Figure 3: Triangular plate and its equivalent rectangular plate (S: simply supported edge)

The conditions $M_r = \theta_r = w = 0$ mean that the bending moment around the tangential axis of the line of point supports, the slope around the normal axis of the line of point supports and the deflection at point support are zero.

The thickness of the actual part of original right triangular plate is expressed as $h$, the thickness of additional part of the equivalent rectangular plate is expressed as $h_t$, and the thickness at a point on the border line between the actual part and the additional part of the equivalent rectangular plate is taken as $(h + h_t)/2$. In this paper, the simply supported edge and clamped edge are denoted by the symbols S and C, respectively.

5 Numerical results

In order to evaluate the accuracy of the numerical so-
lution obtained by the proposed method and investigate the effects of the aspect ratios, modulus ratios and the boundary conditions on the bending of the laminated plates, cross-ply plates and angle-ply plates are analyzed. Four kinds of boundary conditions are considered for rectangular plates and one kind of boundary condition is considered for triangular plates.

**Boundary conditions for rectangular plates are:**

(a) four edges immovable simply supported (SSSS-1):

\[
M_x = \theta_z = \dot{w} = v = u = 0 \quad \text{at} \quad x = 0 \text{ and } x = a \\
M_y = \theta_z = \dot{w} = v = u = 0 \quad \text{at} \quad y = 0 \text{ and } y = b.
\]

(b) four edges movable simply supported condition (SSSS-2):

\[
M_x = \theta_z = \dot{w} = v = N_y = 0 \quad \text{at} \quad x = 0 \text{ and } x = a \\
M_y = \theta_z = \dot{w} = v = N_y = 0 \quad \text{at} \quad y = 0 \text{ and } y = b.
\]

(c) four edges clamped (CCCC)

\[
\theta_z = \dot{w} = v = u = 0 \quad \text{at} \quad x = 0 \text{ and } x = a \\
\theta_z = \dot{w} = v = u = 0 \quad \text{at} \quad y = 0 \text{ and } y = b.
\]

(d) two adjacent edges simply supported and the other two edges clamped (SSCC)

\[
M_x = \theta_z = \dot{w} = v = u = 0 \quad \text{at} \quad x = 0 \\
M_y = \theta_z = \dot{w} = v = u = 0 \quad \text{at} \quad y = 0 \\
\theta_z = \dot{w} = v = u = 0 \quad \text{at} \quad x = a \\
\theta_z = \dot{w} = v = u = 0 \quad \text{at} \quad y = b.
\]

**Boundary conditions for triangular plates are:**

three edges immovable simply supported (SSS-1):

\[
M_x = \theta_z = \dot{w} = v = u = 0 \quad \text{at} \quad x = 0 \\
M_y = \theta_z = \dot{w} = v = u = 0 \quad \text{at} \quad y = 0 \\
M_n = \theta_i = \dot{w_i} = v_i = u_i = 0 \quad \text{at} \quad y = -b/a + b.
\]

In this paper, the transverse deflections at point \((0.5a, 0.5b)\) of a rectangular plate and at point \((1/3a, 1/3b)\) of a triangular plate are solved and expressed as following normalized quantity.

\[
\bar{w} = \frac{w E_1 h_1}{qa^4}
\]

### 5.1 Rectangular plates

Numerical solutions for the deflections of CCC [0°/ 90°] cross-ply rectangular plates under uniform load are given in Table 1. The numerical solutions are obtained by using Richardson's extrapolation formula for the two cases of divisional numbers \(m = n\) of 6 and 8 for the one fourth part of the plate. Table 1 involves the values obtained by Whitney [5] and it shows the good convergency and adequate accuracy of the numerical solutions by the present method.

Figure 4 and Figure 5 give the effects of aspect ratios and ratios of moduli on the deflection of [0°/ 90°] rectangular plates under uniform load. It can be noticed that the deflection will approach a constant with increasing the aspect ratios and the material anisotropy has a great effect on the deflection.

![Figure 4: Deflection of CCC [0°/ 90°] rectangular plates under uniform load vs aspect ratio](image)

**Table 1: Transverse deflection \(\bar{w} \times 10^3\) of CCC [0°/ 90°] cross-ply rectangular plates under uniform load**

\((G_{12} = G_{13} = G_{23} = 0.5, E_2, v_{12} = 0.25, h/a = 0.01)\)

<table>
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<tr>
<th>(b/a)</th>
<th>(E_1 = 40E_2)</th>
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<th>(E_2 = 10E_2)</th>
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<tr>
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<td>(m = 6, 8)</td>
<td>(m = 6, 8)</td>
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Numerical solutions for the deflections of cross-ply square plates under uniform load are given in Table 2. The numerical solutions are obtained by using Richardson’s extrapolation formula for the two cases of divisional numbers \( m (= n) \) of 10 and 12 for the whole part of the plate. Table 2 involves the values obtained by Reddy [6] and it shows the good convergence and adequate accuracy of the numerical solutions by the present method. It is observed that almost no difference of deflections has been found between SSSS-1 and SSSS-2, but great difference exists between SSSS-1 and SCC.

Numerical solutions for the deflections of angle-ply square plates under uniform load are given in Table 3. The numerical solutions are obtained by using Richardson’s extrapolation formula for the two cases of divisional numbers \( m (= n) \) of 8 and 10 for the whole part of the plate. Compared with the results obtained by Kan and Ito [1], the present solutions have good convergence and adequate accuracy. Figure 6 and Figure 7 are used to demonstrate the effects of various boundary conditions, the number of layers and the different angles on the transverse deflection. It can be found the coupling effect is very important for two-layer laminates with large angle \( \theta \) and this effect will decrease with the increasing the number of the layer. Not only the bending boundary conditions but also the in-plane boundary conditions shown in SSSS-1 and

**Table 2: Transverse deflections \( \bar{w} \times 10^3 \) of cross-ply square plates under uniform load**

\((E_1 = 25E_2, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, \nu_{12} = 0.25, h/a = 0.01)\)

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<th>SSSS-1</th>
<th>SSSS-2</th>
<th>SSCC</th>
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</tr>
<tr>
<td>([0^\circ /90^\circ])</td>
<td>10</td>
<td>12</td>
<td>ref.</td>
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</table>

**Table 3: Transverse deflections \( \bar{w} \times 10^3 \) of angle-ply square plates under uniform load**

\((E_1 = 40E_2, G_{12} = G_{13} = G_{23} = E_2, \nu_{12} = 0.25, h/a = 0.01)\)

<table>
<thead>
<tr>
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<th>SSSS-1</th>
<th>SSCC</th>
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</tr>
<tr>
<td></td>
<td>8</td>
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<td>([5^\circ /- 5^\circ])</td>
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<td>4.465</td>
<td>4.450</td>
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</tr>
<tr>
<td>([30^\circ /- 30^\circ])</td>
<td>7.614</td>
<td>7.604</td>
<td>7.585</td>
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<tr>
<td>([45^\circ /- 45^\circ])</td>
<td>7.374</td>
<td>7.515</td>
<td>7.342</td>
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Figure 5: Deflection of CCCC \([0^\circ /90^\circ]\) rectangular plates under uniform load vs moduli ratio
SSSS-2 have a great effect on the results. On this point, the cross-ply plates and angle-ply plates show quite different.

5.2 Triangular plates

Numerical solutions for the deflections of cross-ply and angle-ply triangular plates under uniform load are given in Tables 4 and 5. The numerical solutions are obtained by using Richardson's extrapolation formula for the two cases of divisional numbers \( m = n \) of 8 and 12 for the whole part of the plate. The deflection of isotropic triangular plate is used to show the feasibility of the equivalent rectangular plate and compared with the values obtained by Fletcher [7]. It shows the good convergency and adequate accuracy of the numerical solutions for the isotropic plates by the present method. From Tables 4 and 5, it can be seen that the deflections of all the laminated triangular plates are smaller than that of isotropic triangular plate. For the laminated considered here, the deflections of angle-ply triangular plates are smaller than those of cross-ply triangular plates and the deflection

Table 4: Transverse deflections \( \bar{w} \times 10^4 \) of isotropic and cross-ply triangular plates under uniform load

\( (E_1 = 25E_2, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, \nu_{12} = 0.25, h/a = 0.01) \)

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<th>( 45^\circ/ - 45^\circ )</th>
<th>( 45^\circ/ - 45^\circ )</th>
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<td>5</td>
<td>5.706</td>
<td>6.040</td>
<td>6.470</td>
</tr>
<tr>
<td>6</td>
<td>5.713</td>
<td>6.018</td>
<td>6.410</td>
</tr>
<tr>
<td>7</td>
<td>5.714</td>
<td>6.004</td>
<td>6.376</td>
</tr>
</tbody>
</table>
of $[45°/ -45°]_b$ is the smallest.

6 Conclusions

An approximate method has been proposed for the bending analysis of laminated rectangular and triangular plates under uniform load. The plates considered are cross-ply and angle-ply rectangular and triangular plates. Based on the first-order shear deformation theory, the bending problems of laminated rectangular plates with various boundary conditions which are not restricted to at least a pair of opposite edges simply supported have been solved. The deflections of triangular plates can be obtained by using the equivalent rectangular plate. Compared with the results reported previously, the results obtained by the present method have enough accuracy even with small divisional numbers.

References


Appendix A

\[ F_{1011} = F_{1022} = F_{1039} = F_{1046} = F_{1054} = 1, \quad F_{1066} = D_{12}, \]
\[ F_{1007} = D_{16}, \quad F_{1009} = F_{1108} = B_{12}, \]
\[ F_{1001} = F_{1107} = B_{16}, \quad F_{1002} = D_{22}, \quad F_{1007} = F_{1008} = D_{26}, \]
\[ F_{1120} = B_{22}, \quad F_{1010} = F_{1009} = F_{1107} = F_{1106} = B_{16}, \]
\[ F_{1007} = D_{16}, \quad F_{1006} = k\bar{A}_{44}, \]
\[ F_{1108} = k\bar{A}_{45}, \quad F_{1109} = \bar{A}_{12}, \quad F_{1110} = \bar{A}_{16}, \quad F_{1109} = \bar{A}_{22}, \]
\[ F_{1120} = F_{1109} = \bar{A}_{26}, \quad F_{1110} = \bar{A}_{66}. \]

Appendix B

\[ A_{p01} = \gamma_{p0}, \quad A_{p02} = 0, \quad A_{p03} = \gamma_{p0}, \quad A_{p04} = \gamma_{p0}, \quad A_{p05} = 0, \quad A_{p06} = D_{12}, \]
\[ A_{p07} = D_{16}, \quad A_{p08} = D_{22}, \quad A_{p09} = D_{26}, \]
\[ A_{p10} = D_{44}, \quad A_{p11} = D_{45}, \quad A_{p12} = D_{46}, \quad A_{p13} = 0, \quad A_{p14} = 0, \quad A_{p15} = 0. \]

Appendix C

\[ C_{p01} = \mu \gamma_{p0}, \quad C_{p02} = \mu \gamma_{p0}, \quad C_{p03} = \mu \gamma_{p0}, \quad C_{p04} = \mu \gamma_{p0}, \quad C_{p05} = \mu \gamma_{p0}, \]
\[ C_{p06} = \mu \gamma_{p0}, \quad C_{p07} = \mu \gamma_{p0}, \quad C_{p08} = \mu \gamma_{p0}, \quad C_{p09} = \mu \gamma_{p0}, \quad C_{p10} = \mu \gamma_{p0}, \]
\[ C_{p11} = \mu \gamma_{p0}, \quad C_{p12} = \mu \gamma_{p0}, \quad C_{p13} = \mu \gamma_{p0}, \quad C_{p14} = \mu \gamma_{p0}, \quad C_{p15} = \mu \gamma_{p0}. \]
An analysis of bending problem of laminated rectangular and triangular plates

\[ C_{p12kl} = \mu D_{p12kl}, \quad C_{p13kl} = \mu D_{p13kl} \]

\[ t_{p12}^{-1} \left[ \begin{array}{cc} \rho_{p11} - \mu & -\rho_{p11} \\ -\rho_{p11} & \rho_{p11} - \mu \end{array} \right] \]

\[ \rho_{p11} = \beta, \quad \rho_{p11} = -\mu \]

\[ \beta, \quad \rho_{p11} = \beta, \quad \rho_{p11} = -\mu \]

\[ \rho_{p11} = \beta, \quad \rho_{p11} = -\mu \]

\[ \rho_{p11} = \beta, \quad \rho_{p11} = -\mu \]

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\[ \rho_{p11} = \beta, \quad \rho_{p11} = -\mu \]

\[ \rho_{p11} = \beta, \quad \rho_{p11} = -\mu \]

\[ \text{Appendix C} \]

\[ a_{p12kl} = \sum_{i=0}^{11} \left\{ \sum_{k=0}^{i} \beta_{kl} A_{p12kl} \right\} \]

\[ b_{p12kl} = \sum_{i=0}^{10} \left\{ \sum_{k=0}^{i} \beta_{kl} B_{p12kl} \right\} \]

\[ q_{p12kl} = \sum_{i=0}^{10} \left\{ \sum_{k=0}^{i} \beta_{kl} C_{p12kl} \right\} \]