Stability of an external gravity wave in a stratified basic flow with lateral shear

Author(s)
Tomizuka, Akira

Citation
長崎大学総合環境研究 7(2), p.27-36; 2005

Issue Date
2005-07-30

URL
http://hdl.handle.net/10069/5439

NAOSITE: Nagasaki University’s Academic Output SITE
http://naosite.lb.nagasaki-u.ac.jp
Stability of an external gravity wave in a stratified basic flow with lateral shear

Akira TOMIZUKA

Faculty of Environmental Studies, Nagasaki University

Abstract

The author investigates the stability of an external gravity wave progressing horizontally in an inviscid and incompressible stratified basic flow with lateral shear.

(1) In the model which basic flow has a Helmholtz velocity profile, there exist no neutral solutions contrary to internal gravity waves. Perturbations are always unstable independent of the coefficient of vertical wave mode $m$ or the wave number $k_y$.

(2) In the model which basic flow is composed of unbounded layers with the central shear zone, the stability is subject to the relation of $m$, $k_y$, and width of shearing layer $2h$. The linear shear $\alpha$ affects only a damping or a growing factor.

(a) In the case of $k_y \geq m$, there exist a neutrally stable solution $k_y = \sqrt{m^2 + (0.639232/h)^2}$. The region $k_y > \sqrt{m^2 + (0.639232/h)^2}$ is stable, and the region $m \leq k_y < \sqrt{m^2 + (0.639232/h)^2}$ is unstable.

(b) In the case of $m > k_y$, there exist two traveling modes, a damping mode and a growing mode. So perturbations are unstable. And if $m \gg k_y$, they can be stable.

Keywords: stability, external gravity wave, lateral shear
1. Introduction

The property of instability in stratified flows is important especially for flows in the atmosphere and ocean. For internal gravity waves, the interactions of them with basic flows have already been fully studied. For example, BOOKER & BETHERTON (1967) clarified internal gravity waves are thoroughly absorbed into the basic flow at the critical level for Richardson number $R_j > 1.0$ and are partially absorbed for $0.25 < R_j < 1.0$. JONES (1968) found out the over-reflexion, which the reflected energy flux is larger than the incident one because waves get energy from the basic flow. Also JONES (1972) investigated the ducting property in the basic flow with three stratified layers, and showed the wave is less propagating with growth of shear. LINDZEN (1974) pointed out the existence of neutrally stable internal gravity waves in the basic flow with a Helmholtz velocity profile for small wave number. MAJDA & SHEFTER (2000) studied nonlinear instability of the stratified flows through numerical simulations and showed the nonlinear development of perturbations can lead to significant growth of kinetic energy without significant transfer to the potential energy above the critical value.

In previous papers, the author investigates the interaction between an external gravity wave and a basic shear flow in the sea (MATSUSHIMA et al., 1995). One of the results is that perturbations interact with the basic flow fairly away from the critical level. This work in this paper discusses the stability of an external gravity wave in several models of stratified shear flow.

2. Governing equations

Consider the basic steady flow of fluids progressing horizontally on the horizontal surface. The Y-axis is taken parallel to the basic flow, X-axis normal to the basic flow on the surface, and Z-axis vertical upward to the surface. We express the pressure and velocity of the basic flow with a lateral shear as $P(z)$ and $V = (V_x, 0, 0)$. We suppose the disturbance traveling in the basic flow, whose pressure and velocity are $p'$ and $v' = u_i + v_j + w_k$.

We represent total current and total pressure with prime as follows:

$$\bar{v}' = \bar{V} + \bar{v},$$

$$p' = P(z) + p.$$  \hspace{1cm} (1)

Equations of motion and equation of continuity are given as follows:

$$\left( \frac{\partial}{\partial t} + \bar{v}' \cdot \nabla \right) \bar{v}' = -\nabla \frac{p'}{p} - g \kappa + n \Delta \bar{v}',$$

$$\nabla \cdot \bar{v}' = 0,$$  \hspace{1cm} (3)

where $\rho$ is the density of the incompressible fluid, $\nu$ is the coefficient of kinematic viscosity and $g$ is the gravitational acceleration.

Now we consider the relations

$$\nabla \frac{p'}{p} = \nabla \frac{P(z) + p}{p} = \nabla \left( \frac{p}{p} \frac{dP(z)}{dz} \right) = \nabla \frac{p}{p} - \frac{g}{\rho},$$

and
Stability of an external gravity wave in a stratified basic flow with lateral shear

\[(\vec{v}' \cdot \nabla)\vec{v}' = \left(\vec{v}' \cdot \nabla + V \frac{\partial}{\partial y}\right)\vec{v}' = (\vec{v}' \cdot \nabla)\vec{v}' + V \frac{\partial}{\partial y} \vec{v}' + \frac{dV}{dx} \vec{f}\]

\[= V \frac{\partial}{\partial y} \vec{v}' + \frac{dV}{dx} \vec{f}\]

Equations (3) and (4) are written as follows:

\[\left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial y}\right)\vec{v}' + \frac{dV}{dx} \vec{f} = -\nabla \frac{\rho}{\rho} + \nu \Delta \vec{v}', \quad (5)\]

\[\nabla \cdot \vec{v}' = \nabla \cdot (\vec{V} + \vec{v}') = \nabla \cdot \vec{v}' + \frac{\partial V}{\partial y} = \nabla \cdot \vec{v} = 0. \quad (6)\]

In addition, when the inviscid fluid is assumed, the equations for such perturbations are

\[\frac{\partial u}{\partial t} + V \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial \rho}{\partial x}, \quad (7)\]

\[\frac{\partial w}{\partial t} + V \frac{\partial w}{\partial y} + \frac{dV}{dx} = -\frac{1}{\rho} \frac{\partial \rho}{\partial y}, \quad (8)\]

\[\frac{\partial w}{\partial t} + V \frac{\partial w}{\partial y} = -\frac{1}{\rho} \frac{\partial \rho}{\partial z}, \quad (9)\]

and

\[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (10)\]

We consider perturbations of the form as follows:

\[\vec{v}(x,y,z) = \vec{v}(x)e^{im(x-ct-z)}\]

\[\rho(x,y,z) = \rho(x)e^{im(x-ct-z)} , \quad (11)\]

in which it is understood that the real parts of these expressions must be taken to obtain physical quantities. Also we shall assume \(m > 0\) as the wave becomes extinct at \(z \rightarrow -\infty\).

We express the phase velocity as \(c = c_i + ic\). The instability is subject to the imaginary part \(c_i\). If \(c_i = 0\) perturbations are neutrally stable and then traveling modes, and if \(c_i > 0\) perturbations are stable and then damping modes because \(e^{ik_i t} = e^{-k_i t} \rightarrow 0 (t \rightarrow \infty)\). And if \(c_i < 0\) perturbations are unstable and then growing modes because \(e^{-k_i t} \rightarrow \infty (t \rightarrow \infty)\). If there exists at least only one growing mode, perturbations are always unstable.

On substituting the expression (11) into equations (7)-(10), we obtain the system of differential equations:

\[ik_j(c-V)\vec{u} = -\frac{1}{\rho} \frac{d\rho}{dx}, \quad (12)\]

\[ik_j(c-V)\vec{u} + \frac{dV}{dx} \vec{u} = ik_j \frac{1}{\rho} \vec{p}, \quad (13)\]

\[ik_j(c-V)\vec{w} = -\frac{m}{\rho} \vec{p}, \quad (14)\]
and
\[
\frac{d \nu}{d x} - i k \nu + m \nu = 0. \tag{15}
\]

We reduce from these equations to the following equation of second order for \( \bar{p} \):
\[
\frac{d^2 \bar{p}}{dx^2} + \frac{2}{c-V} \frac{dV}{dx} \frac{d \bar{p}}{dx} + \left\{ m^2 - k^2 \right\} \bar{p} = 0. \tag{16}
\]

In addition, the velocity components of perturbations are shown as the function of \( \bar{p} \):
\[
\bar{u} = \frac{i}{k_j (c-V) \rho} \frac{d \bar{p}}{dx}, \tag{17}
\]
\[
\bar{v} = -\frac{1}{k_j^2 (c-V)^2} \frac{dV}{dx} \frac{d \bar{p}}{dx} + \frac{1}{(c-V) \rho} \bar{p}, \tag{18}
\]
\[
\bar{w} = \frac{im}{k_j (c-V) \rho} \bar{p}. \tag{19}
\]

3. Velocity profiles and stability

(1) a Helmholtz profile

![Helmholtz profile diagram](image)

**Figure 1** The basic flow with a Helmholtz velocity profile.
Stability of an external gravity wave in a stratified basic flow with lateral shear

First we consider the basic flow with a Helmholtz velocity profile shown as Fig. 1, which is discontinuous at \( x = 0 \) and \( V = V_0 \) for \( x > 0 \), \( V = -V_0 \) for \( x < 0 \). In this case, equation (16) reduces to

\[
\frac{d^2 \tilde{p}}{dx^2} + \left\{ m^2 - k_y^2 \right\} \tilde{p} = 0, \tag{20}
\]

because \( \frac{dV}{dx} = 0 \). At \( x = 0 \) we require the matching condition as the continuity of \( \tilde{p} \). And for discontinuity of \( V \), another matching condition is not the continuity of \( \tilde{u} \) but \( \tilde{u}(c-V) \) (DRAZIN & REID, 1981).

We obtain the general solution of equation (20) as

\[
\tilde{p}(x) = Ae^{i\alpha x} + Be^{-i\alpha x},
\]

where \( \Omega^2 = k_y^2 - m^2 \). In order that perturbations disappear at \( x \to \pm \infty \) we can take

\[
\tilde{p}(x) = \begin{cases} 
Ae^{i\alpha x} & (x < 0) \\
Be^{-i\alpha x} & (x > 0)
\end{cases}
\tag{21}
\]

From equation (17) we have \( \tilde{u}(x) \) as follows:

\[
\tilde{u}(x) = \frac{ix}{k_y(c+V_0)\rho} Ae^{i\alpha x} (x < 0) - \frac{ix}{k_y(c-V_0)\rho} Be^{-i\alpha x} (x > 0)
\tag{22}
\]

Continuity of \( \tilde{p}(x) \) at \( x = 0 \) requires \( A = B \). So the matching condition for \( \tilde{u}(c-V) \) at \( x = 0 \) leads to the relation:

\[
c^2 + V_0^2 = 0. \tag{23}
\]

We can take the solution

\[
c = \pm iV_0. \tag{24}
\]

The perturbations have a damping mode \( c = iV_0 \) and a growing mode \( c = -iV_0 \). So the instability appears for the latter. These results are independent of magnitude of \( m \) or \( k_y \).

There exist no neutral solutions that \( c = 0 \) for external gravity waves. On the contrary, a neutral solution was obtained in the internal gravity waves propagating upward (LINDZEN, 1974).
unbounded layers with the central shear zone

Consider next the basic flow with a shear layer shown in Fig. 2. This profile is made by putting the shear layer of width $2h$ into the Helmholtz profile at $x = 0$. We express the profile as

$$V(x) = \begin{cases} 
-V_0 & x < -h \\
\alpha x & |x| \leq h \\
V_0 & x > h 
\end{cases}$$

where $\alpha = V_0 / h$ is a shear of the flow.

In the shear layer of $|x| \leq h$ equation (16) reduces to

$$\frac{d^2 \tilde{p}}{dx^2} + \frac{2}{a-x} \frac{d \tilde{p}}{dx} - \Omega^2 \tilde{p} = 0,$$

where $a = c / \alpha$, $\Omega^2 = \kappa^2 - \frac{m^2}{h^2}$.

Equation (25) is solved analytically and we obtained independent solution each other as

$$\tilde{p}_1(x) = \frac{(a-x)\Omega - 1}{4\Omega^2} e^{\Omega(a-x)},$$

$$\tilde{p}_2(x) = \frac{3}{2\Omega} \left[ \left(1 + (a-x)\Omega \right) e^{\Omega(a-x)} - \left(1 - (a-x)\Omega \right) e^{\Omega(a+x)} \right].$$

Then $\bar{u}$ becomes

$$\bar{u}_1(x) = -\frac{i}{4k,\alpha \rho \Omega} e^{\Omega(a-x)},$$

$$\bar{u}_2(x) = \frac{3i}{2k,\alpha \rho \Omega} \left[ -e^{\Omega(a-x)} + e^{-\Omega(a-x)} \right],$$

respectively.

The other side, we have the solution in the layers $|x| > h$

$$\tilde{p}(x) = \begin{cases} 
e^{\alpha x} & x < -h \\
e^{-\alpha x} & x < h 
\end{cases}$$

in order perturbations disappear at $x \to \pm \infty$.

So we express the solution as follows:

Figure 2 The basic flow, which is composed of unbounded layers with the central shear zone.
Stability of an external gravity wave in a stratified basic flow with lateral shear

\[ \bar{p}(x) = \begin{cases} Ae^{\alpha x} & x < -h \\ B\bar{h}(x) + C\bar{p}_2(x) & |x(x)| \leq h \\ De^{\alpha x} & x > h \end{cases} \]  

(30)

On applying the matching conditions of \( \bar{p}(x) \) and \( \bar{u}(x) \) at \( x = -h \) we obtain the next relation:

\[-B\{2(a + h)\Omega - 1\} = 6C\{2(a + h)\Omega - 1 + 2e^{-2Q(a+h)}\}.\]  

(31)

And also at \( x = h \),

\[ B = 6C\left[2(a - h)\Omega + 1\right]e^{-2Q(a-h)} - 1.\]  

(32)

From equations (31) and (32) we obtain the relation for \( c^2 \):

\[ 4c^2 = \frac{\alpha^2}{Q^2}\left((2hQ - 1)^2 - e^{-4hQ}\right).\]  

(33)

Figure 3 shows \( c^2 \) as a function of \( h\Omega \) normalized by \( h^2\Omega^2 \). If \( c^2 = 0 \) is solved, we obtain \( h\Omega = 0.639232 \cdot \). It is obvious \( c^2 < 0 \) for \( h\Omega < 0.639232 \). The stability is subject to the sign of \( c^2 \).

\[ \frac{c^2}{h^2\Omega^2} \quad (0.639232, 0) \]

Figure 3 The curve of \( c^2 \) as a function of \( h\Omega \) normalized by \( h^2\Omega^2 \), which determines the stability of perturbations.
**Akira TOMIZUKA**

(a) **in the case of** \( \Omega^2 = k_y^2 - m^2 \geq 0 \)

In this case, \( c \) is zero for \( \Omega = 0.639232/h \), pure imaginary for \( 0 \leq \Omega < 0.639232/h \), and real for \( \Omega > 0.639232/h \). So the stability is as follows:

(i) **neutrally stable** for \( k_y = \sqrt{m^2 + (0.639232/h)^2} \)

(ii) **stable** for \( k_y > \sqrt{m^2 + (0.639232/h)^2} \)

(iii) **unstable** for \( m \leq k_y < \sqrt{m^2 + (0.639232/h)^2} \)

Figure 4 shows \( c^2 \) with \( h = 0.5 \) as functions of \( k_y \) and \( m \) normalized by \( \alpha^2/4 \). The \( m > k_y \) is the zone where \( c \) is complex numbers because \( \Omega \) is imaginary.

(b) **in the case of** \( \Omega^2 = k_y^2 - m^2 < 0 \)

We let \( i\Gamma = \Omega \) where \( \Gamma \) is real and positive. From equation (33), we obtain the relation for \( c_r \) and \( c_i \):

\[
\begin{align*}
\begin{cases}
c_r^2 - c_i^2 &= \{4h^2\Gamma^2 + \cos(4h\Gamma) - 1\} \frac{\alpha^2}{4\Gamma^2} \\
2c_r c_i &= \{4h\Gamma - \sin(4h\Gamma)\} \frac{\alpha^2}{4\Gamma^2}
\end{cases}
\end{align*}
\]

(34)

In the case of \( h\Gamma \ll 1 \), neglecting more than 3\(^{rd}\) order terms, equation (34) reduces to

\[
\begin{align*}
\begin{cases}
c_r^2 - c_i^2 &= -h^2 \alpha^2 \\
2c_r c_i &= 0
\end{cases}
\end{align*}
\]

(35)

This relation must lead to \( c_r = 0 \) and \( c_i = \pm h\alpha \), so perturbations are unstable. This result agrees with the

![Figure 4](image-url) The outlook of \( c^2 \) where \( m \leq k_y \) with \( h = 0.5 \), normalized by \( \alpha^2/4 \). The region \( m > k_y \) is the zone where \( c \) is complex numbers because \( \Omega \) is imaginary. Perturbations are unstable for negative \( c^2 \) region which is enclosed by \( k_y = \sqrt{m^2 + (0.639232/h)^2} \) and \( m = k_y \).
Stability of an external gravity wave in a stratified basic flow with lateral shear

...conclusion in the case of (a) at \( m = k_y \).

In the case of \( h\Gamma >> 1 \), on the contrary, equation (34) reduces to

\[
\begin{align*}
\left\{ \begin{array}{l}
\epsilon_1^2 - \epsilon_2^2 = h^2 \alpha^2 \\
\epsilon_1 \epsilon_2 = 0
\end{array} \right.
\end{align*}
\]

(36)

This relation must lead to \( \epsilon_1 = 0 \) and \( \epsilon_2 = \pm h\alpha \), so perturbations are stable.

Now we get the values of \( \epsilon_r \) and \( \epsilon_i \) for some \( h\Gamma \)'s. We have \( \epsilon_r = 0.9779 \) and \( \epsilon_i = 0.6080 \) at \( h\Gamma = 1 \), also \( \epsilon_r = 0.9887 \) and \( \epsilon_i = 0.2216 \) at \( h\Gamma = 2 \) where the negative values are omitted.

More precisely, \( \epsilon_r \) and \( \epsilon_i \) are shown as in Fig. 5 which are normalized by \( h\alpha \). Generally, in the case of \( m > k_y \), there must exist two traveling modes, a damping mode and a growing mode from the above discussion and perturbations are unstable. And also the instability decreases with the magnitude of \( m \). Furthermore if \( m \gg k_y \) perturbations can be stable.

The linear shear \( \alpha \) affects only a damping or a growing factor.

![Figure 5](image)

**Figure 5** The comparison of a real component \( \epsilon_r \) and an imaginary one \( \epsilon_i \), which are normalized by \( h\alpha \).
4. Conclusion

We investigate the stability for external gravity waves in an inviscid and incompressible stratified basic flow with lateral shear. The results are as follows:

(I) a Helmholtz profile

The basic flow with a Helmholtz velocity profile has discontinuity at $x=0$ and $V=V_0$ for $x>0$, $V=-V_0$ for $x<0$. There exist no neutral solutions contrary to internal gravity waves. Perturbations are always unstable independent of vertical wave factor $m$ or wave number $k_y$.

(II) unbounded layers with the central shear zone

This profile is made by putting the shear layer of width $2h$ into the Helmholtz profile at $x=0$. The stability is subject to the relation of $h$, $m$ and $k_y$ as follows:

\[
\begin{align*}
  k_y &> \sqrt{m^2 + (0.639232/h)^2} & \text{stable} \\
  k_y &= \sqrt{m^2 + (0.639232/h)^2} & \text{neutral stable} \\
  m \leq k_y < \sqrt{m^2 + (0.639232/h)^2} & \text{unstable} \\
  k_y < m & \text{unstable} \\
  k_y < < m & \text{stable}
\end{align*}
\]

In the case of $k_y < m$, there exist two traveling modes, a damping mode and a growing mode.

References


