Optimal $L^p$ Estimates for the $\bar{\partial}$ Equation on Real Ellipsoids

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abstract

Let $D$ be a real ellipsoid in $C^n$. In this paper we give optimal $L^p$ estimates for solutions of the $\bar{\partial}$-problem on $D$.

1. Introduction

Range[5] obtained Hölder estimates for solutions of the equation $\bar{\partial}u = f$ on complex ellipsoids when $f$ is a $(0, 1)$-form. Chen-Krantz-Ma[1] obtained optimal $L^p$ estimates for solutions of the equation $\bar{\partial}u = f$ on complex ellipsoids when $f$ is a $(0, 1)$-form. On the other hand, Ho[4] obtained Hölder estimates for solutions of the equation $\bar{\partial}u = f$ on complex ellipsoids when $f$ is a $(0, q)$-form. Further, Diederich-Fornaess-Wiegerinck[2] obtained Hölder estimates for solutions of $\bar{\partial}$ on real ellipsoids. Fleron[3] studied Hölder estimates for solutions of the $\bar{\partial}$ problem on the complement of real or complex ellipsoids. In this paper we study the optimal $L^p$ estimates for solutions of the $\bar{\partial}$-equation on real ellipsoids.

2. Solutions of the $\bar{\partial}$ equation on real ellipsoids

Let $l_1, \ldots, l_s, m_1, \ldots, m_s$ be positive even integers and let $D$ be the real ellipsoid

$$D = \{z \in C^n : r(z) < 0\},$$

where

$$r(z) = \sum_{k=1}^{s} (x_k^l + y_k^m) - 1, \quad z_k = x_k + iy_k.$$

We set

$$m = \max_{1 \leq k \leq s} \min(l_k, m_k).$$

We may assume $m_k \leq l_k$. We set $\phi_k(x) = x_k^l$, $\phi_k(y) = y_m^m$. For some positive constant $\gamma$ and $\xi_j = \xi_j + i\eta_j$ we set

$$P_j(\xi, z) = -2 \frac{\partial^{r}}{\partial \xi_j} (\xi_j) + \gamma (\phi_j^l(\eta_j) - \phi_j^m(\xi_j)) (z_j - \xi_j) + (z_j - \xi_j)^{m_j - 1}$$
and

\[ \Phi(\zeta, z) = \sum_{j=1}^{n} P_j(\zeta, z)(z_j - \zeta_j) \quad \text{for} \quad z, \zeta \in \overline{D}. \]

If we choose \( \gamma \) small enough, then we have for some positive constant \( c \) (Diederich-Fornaess-Wiegerinck[2])

\[
(1) \quad -r(\zeta) + r(z) + \Re \Phi(\zeta, z) \geq c \sum_{j=1}^{n} \left\{ |\phi_j'(\zeta_j) + \phi_j'(\eta_j)| |z_j - \zeta_j|^2 + |z_j - \zeta_j|^m \right\}
\]

for \( (\zeta, z) \in \overline{D} \times \overline{D} \).

Define

\[
\beta = |z - \zeta|^2, \quad B(\zeta, z) = \frac{\partial_i \beta}{\beta}, \quad W(\zeta, z) = \sum_{j=1}^{n} P_j(\zeta, z) \, d_{\xi_j}^*,
\]

\[
\hat{W}(\zeta, z) = \lambda W(\zeta, z) + (1 - \lambda) B(\zeta, z),
\]

\[
\Omega_\epsilon(W) = c_\epsilon W \wedge (\overline{\partial} \epsilon W)^{n+q-1} \wedge (\overline{\partial} \epsilon W)^q,
\]

where

\[
c_\epsilon = \left( -1 \right)^{q(e-1)/2} \left( \frac{n-1}{\pi} \right)^{q/2}
\]

is a numerical constant. \( \Omega_\epsilon(\hat{W}) \) is defined in the same way, with \( \hat{W} \) instead of \( W \). We define \( K_\epsilon = \Omega_\epsilon(B) \). Then we have the following (cf. Range[6]):

**Lemmas.** Let \( f \) be a \( C^1(0, q) \)-form in \( \overline{D} \). Define

\[
T^w_{\epsilon} f = \int_{\partial D \times [0,1]} f \wedge \Omega_{\epsilon-1}(\hat{W}) - \int_D f \wedge K_{\epsilon-1}.
\]

Then \( u = T^w_{\epsilon} f \) is a solution of the equation \( \overline{\partial} u = f \).

### 3. Optimal \( L^p \) estimates

Using the solution of the \( \overline{\partial} \) equation in lemma 1 and (1), we have the following (Show[7] obtained the optimal \( L^p \) estimate for solutions of the \( \overline{\partial} \)-problem on \( D \)):

**Theorem 1.** For every \( \overline{\partial} \)-closed \( (0, q) \)-form \( f \) with coefficients in \( L^p(D) \), there exists a \( (0, q - 1) \)-form \( u \) on \( D \) such that \( \overline{\partial} u = f \) and \( u \) satisfies the following estimates:

(i) If \( p = 1 \), then \( \| u \|_{L^\gamma(D)} \leq c \| f \|_{L^1(D)} \), where \( \gamma = \frac{mn+2}{mn+1} \).

(ii) If \( 1 < p < mn+2 \), then \( \| u \|_{L^s(D)} \leq c \| f \|_{L^p(D)} \), where \( s < q_0 \) and \( q_0 \) satisfies \( \frac{1}{q_0} = \frac{1}{p} - \frac{1}{mn+2} \).

(iii) If \( p = mn+2 \), then \( \| u \|_{L^s(D)} \leq c \| f \|_{L^p(D)} \) for all \( s < \infty \).

(iv) If \( p > mn+2 \), then \( \| u \|_{A_{a}(D)} \leq c \| f \|_{L^p(D)} \), where \( a = \frac{1}{m} - \frac{2}{m} \frac{1}{p} \).

**Proof.** Define

\[
J_1(f) = \int_{\partial D \times [0,1]} f \wedge \Omega_{\epsilon-1}(\hat{W}), \quad J_2(f) = \int_D f \wedge K_{\epsilon-1}.
\]
Then $I_j(f)$ is a linear combination of $I_j(0 \leq j \leq n-q-1)$:

$$I_j(z) = \int_{\partial D} f(\xi) \wedge \partial_i \beta \wedge P \wedge (\tilde{\partial}_j P) \wedge (\tilde{\partial}_i \partial_j \beta)^{s-q-j-1} \wedge (\tilde{\partial}_i \partial_j \beta)^{q-1} \Phi(\xi, z)^{s-j-1} \beta(\xi, z)^{s-j-1}$$

where $P = \sum_{i=1}^n P_i d\gamma_i$.

Define

$$\Phi(\xi, z) = \Phi(\xi, z) - r(\xi), \quad b(\xi, z) = |\xi - z|^2 + r(\xi) r(z).$$

Then we have

$$I_j(z) = \int_{\partial D} f(\xi) \wedge \tilde{\partial}_j \left( \frac{\partial_i \beta \wedge P \wedge (\tilde{\partial}_j P) \wedge (\tilde{\partial}_i \partial_j \beta)^{s-q-j-1} \wedge (\tilde{\partial}_i \partial_j \beta)^{q-1}}{\Phi(\xi, z)^{s-j-1} \beta(\xi, z)^{s-j-1}} \right),$$

where $\tilde{\partial}_j$ denotes the tangential component of $\tilde{\partial}$. For a neighborhood $U$ of some boundary point, we may choose a system of local coordinates $t = (t_1, \cdots, t_{2n})$ in such a way that

\begin{align*}
    t_k &= t_{2k-1} + it_{2k} = z_k - \xi_k \quad (k = 1, \cdots, n-1) \\
    t_{2k-1} &= \text{Im} \Phi(\xi, z) \\
    t_{2k} &= r(\xi) - r(z).
\end{align*}

For $a > 1$ and $s (0 \leq s \leq n-q-1)$ we set

$$\mathbf{A} = \int_{D \cap U} \frac{\left| (\tilde{\partial}_j P(\xi)) \right|^s}{\left| \Phi(\xi, z) \right|^{|s|+2} b(\xi, z)^{\left(2s-2s-3/2 \right)^2}} \, d\mu(\xi).$$

Define

$$A_j = |x_j - t_{2j-1}|^{s-j-2} + |y_j - t_{2j}|^{s-j-2} + (|x_j - t_{2j-1}|^{s-j-3} + |y_j - t_{2j}|^{s-j-3}) |t_j|$$

$$B_j = (|x_j - t_{2j-1}|^{s-j-2} + |y_j - t_{2j}|^{s-j-2}) |t_j|^2 + |t|^{|s|},$$

We define $t' = (t_1', \cdots, t_{n-1}')$, $t' = (t_{2n+1}', \cdots, t_{2n-2}')$, $t = (t_1, t_{2n-1}, t_{2n})$. Then we have

$$I_j^s(z) \leq \int_{|t| \leq 1} \frac{\sum_{i=0}^{\infty} A_i}{(t_{2i-1}^2 + t_{2i})^{1/2} + \sum_{k=1}^{n-1} B_k} \left| \frac{dt}{|t|^{2s+2}} \right| |t|^{|s|-(2s-2s-3)|} \, dt$$

$$\leq c \int_0^1 \left( |t|^{|s|} \right) ^{1/2} \left| \frac{dt}{|t|^{2s+2}} \right| |t|^{|s|-(2s-2s-3)|} < \infty,$$

provided that

$$a < m \frac{(m+2) + 2n-2s-2}{m (m+2) + 2n-2s-3} = a_*.$$

Since

$$a_0 > a_1 > \cdots > a_{n-2} \leq \frac{mn+2}{mn+1}$$
we have proved that

\[ \int_D |K(\xi, z)|^a \, d\mu(\xi) < M_1 \quad \text{uniformly} \quad z \in D, \]

where \( a \) is any number such that

\[ 1 < a < \frac{mn+2}{mn+1}. \]

Similarly we have

\[ \int_D |K(\xi, z)|^a \, d\mu(\xi) < M_2 \quad \text{uniformly} \quad \xi \in D. \]

Therefore we have proved (i), (ii) and (iii) of theorem 1. The worst term we need to estimate for \( \text{grad} \cdot I_j(\xi, z) \) is given by

\[ I(\xi, z) = \frac{|(\partial^2 \bar{r}(\xi))^{s-2}|}{|\Phi(\xi, z)|^{s-1}|z-z'|}. \]

Let \( t \) be conjugate to \( p \). Then

\[ \left( \int_D |I(\xi, z)|^t \, d\mu(\xi) \right)^\frac{1}{t} \leq c \left( \int_D \frac{|(\partial^2 \bar{r}(\xi))^{s-2}|}{|\Phi(\xi, z)|^{t(s+1)}|z-z'|} \, d\mu(\xi) \right)^\frac{1}{t} \]

\[ \leq \frac{c}{|r(\xi)|^{1-\frac{1}{t} + \frac{1}{t(s+1)}}}. \]

By the Hölder inequality, we have

\[ \int_D |\text{grad} \cdot I_j(\xi, z)| \, d\mu(\xi) \leq c \| f \|_{L_p} \| r(z) \|^{1-\frac{1}{t} + \frac{1}{t(s+1)}}. \]

This proves (iv).

Let \( 1 \leq q \leq n - 1 \). Let \( \Delta^q \) be the maximal order of contact of the boundary of the real ellipsoid \( D \) with \( q \)-dimensional complex linear subspaces. Suppose that \( l_i \geq m_i \) \( (j = 1, \ldots, n) \) and \( m_1 \leq m_2 \leq \cdots \leq m_n \). Then \( \Delta^q = m_{n-q+1} \). Using the method of Ho[4], theorem 1 is improved a bit. Now we define

\[ \tilde{P}(\xi_i, z_i) = \begin{cases} P_i(\xi_i, z_i) & (1 \leq i \leq n - q + 1) \\ P_i(\xi_i, z_i) + \bar{\zeta}_i - \zeta_i & (n - q + 2 \leq i \leq n) \end{cases}. \]

Define

\[ \Phi(\xi, z) = \sum_{i=1}^n \tilde{P}(\xi_i, z_i)(\xi_i - z_i), \]

\[ \zeta_i - z_j = v_j, \quad \xi_i = \xi_i + i\eta_j \quad (j = 1, \ldots, n) \]

and

\[ v' = (v_1, \ldots, v_{n-q+1}), \quad v'' = (v_{n-q+2}, \ldots, v_n) \]

Then we have the following:
LEMMA 2. Let $m = A^q$. Then there exists a positive constant $c$ such that

$$-r(\xi) + r(z) + \Re \Phi(\xi, z) \geq c \left\{ \sum_{j=1}^{n-1} \left( \phi_j(\xi_j) + \phi_j(\eta_j) \right) |z_j - \xi_j|^2 + |v|^m + |v^{-1}|^2 \right\},$$

for $(\xi, z) \in \overline{D} \times \overline{D}$.

Using the argument of the proof of theorem 1, we have the following:

THEOREM 2. Let $m = A^q$ and $p \geq 1$. For every $\overline{\partial}$-closed $(0, q)$-form $f$ with coefficients in $L^p(D)$, there exists a $(0, q-1)$-form $u$ on $D$ such that $\overline{\partial} u = f$ and $u$ satisfies the following estimates:

(i) If $p = 1$, then $\|u\|_{L^{1,\infty}(D)} \leq c \|f\|_{L^1(D)}$, where $\gamma = \frac{mn+2}{mn+1}$ and $\varepsilon$ is any small number.

(ii) If $1 < p < mn+2$, then $\|u\|_{L^{1,\varepsilon}(D)} \leq c \|f\|_{L^p(D)}$, where $\varepsilon < q_0$ and $q_0$ satisfies $\frac{1}{q_0} = \frac{1}{p} - \frac{1}{mn+2}$.

(iii) If $p = mn+2$, then $\|u\|_{L^{1,\varepsilon}(D)} \leq c \|f\|_{L^p(D)}$ for all $\varepsilon < \infty$.

(iv) If $p > mn+2$, then $\|u\|_{L^{1,\varepsilon}(D)} \leq c \|f\|_{L^p(D)}$, where $\alpha = \frac{1}{m} - \left( n + \frac{2}{m} \right) \frac{1}{p}$.

References


