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On Rotation Matrices of given Axes and Angles and the Group Structure on $SO(3)$

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Abstract

We treat rotation matrices of given axes and angles in the space $\mathbb{R}^3 = \text{Im}\mathbb{H}$ of pure imaginary quaternions. We give a product formula of rotation matrices of given axes vectors and so explain the group structure on $SO(3) \simeq \mathbb{R}P^3$ from the viewpoint of axes and angles.

1 Introduction

We give the matrix expression $g(\theta; u) \in SO(3)$ of rotation in $\mathbb{R}^3$ of given axis $u \in \mathbb{R}^3$, $|u| = 1$ and angle $\theta$ by using the adjoint representation $\text{Ad}: S^3 = Sp(1) \longrightarrow SO(3)$, as the following form:

$$g(\theta; u) = g(\theta u) = \text{Ad} \left( \exp \frac{\theta}{2} u \right)$$

where $u \in \mathbb{R}^3$ is identified with a quaternion in $\text{Im}\mathbb{H}$ and $\theta u \in \mathbb{R}^3$ is called the axis vector of the rotation. $g(\theta; u)$ is to rotate clockwise around the axis $u$ with angle $\theta$. The description is classically known as the Cayley-Klein parameter, and is equivalent to that given by the adjoint representation of $SU(2)$. We next give the product formula:

$$g(\theta_1; u_1)g(\theta_2; u_2) = g(\theta_3; u_3)$$

and so look closely at the group structure in $SO(3) = \mathbb{R}P^3$ which is a closed ball of radius $\pi$ in $\mathbb{R}^3$ whose antipodal points in the boundary are identified.
2 Description of Rotational Transformation by Quaternions

We identify the set \( \text{Im} \mathbb{H} \) of all pure imaginary quaternions with the real 3-dimensional space \( \mathbb{R}^3 \) by a linear isomorphism over \( \mathbb{R} \):

\[
\begin{align*}
\mathbb{R}^3 & \sim \text{Im} \mathbb{H} \\
\begin{pmatrix} a \\ b \\ c \end{pmatrix} & \mapsto ai + bj + ck
\end{align*}
\]  

(1)

Let \( x = x_1i + x_2j + x_3k, \ y = y_1i + y_2j + y_3k \in \text{Im} \mathbb{H} \). Define an inner product in \( \text{Im} \mathbb{H} \) by

\[
(x, y) = x_1y_1 + x_2y_2 + x_3y_3.
\]

Then identification (1) is an isomorphism of Euclidean spaces.

Let \( S^3 = Sp(1) = \{ \rho \in \mathbb{H} | |\rho| = 1 \} \). For \( \rho \in S^3 \), we denote the adjoint representation of \( S^3 = Sp(1) \) by \( F_\rho \):

\[
F_\rho = \text{Ad}_\rho : x \mapsto \rho x \rho^{-1}, \ \text{Im} \mathbb{H} \to \text{Im} \mathbb{H}.
\]

(2)

For any \( u \in \text{Im} \mathbb{H}, \ |u| = 1 \), we have \( u^2 = -1 \). Hence the exponential is given by

\[
e^{\theta u} = \cos \theta + u \sin \theta, \ \theta \in \mathbb{R}.
\]

The exponential map \( \exp : \text{Im} \mathbb{H} \to S^3 \) is then surjective. We show that

1. The sequence: \( 1 \to \{ \pm 1 \} \to S^3 \xrightarrow{F} SO(3) \to 1 \) is exact,

2. If \( \rho = e^{\theta u}(u \in \text{Im} \mathbb{H}, |u| = 1) \) then \( F_\rho \) has \( u \) as axis and \( \theta \) as angle.

2.1 \( F(S^3) = SO(3) \) and \( \text{Ker} \ F = \{ \pm 1 \} \)

\( \text{Ker} \ F = \{ \pm 1 \} \) is a consequence of center(\( \mathbb{H} \))=\( \mathbb{R} \) because \( \mathbb{R} \cap S^3 = \{ \pm 1 \} \). The formula

\[
(x, y) = \frac{1}{2}(xy + yx)
\]

(3)

shows not changing an inner product by \( F_\rho \), i.e.,

\[
(F_\rho(x), F_\rho(y)) = (x, y).
\]

So \( F(S^3) \subset O(3) \). The map \( \rho \mapsto \det F_\rho \) is a continuous map from a connected \( S^3 \) to \( \{ \pm 1 \} \), we have \( \det F_\rho = +1 \) and so \( F(S^3) \subset SO(3) \). Since \( \dim S^3 = \dim SO(3) = 3 \) and \( F \) is a continuous homomorphism between connected groups with discrete kernel, we know that \( F(S^3) = SO(3) \).
2.2 Axes and Angles

We show that $F_p (\rho = e^{\theta u/2})$ has $u$ as axis and $\theta$ as clockwise angle of rotation. We use the formula

$$F_p(x) = x \cos \theta + (u \times x) \sin \theta + (u, x) u (1 - \cos \theta) \quad (4)$$

where $u \times x$ is an outer product given by

$$x \times y = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \\ i & j & k \end{vmatrix}.$$  

$F_p$ has $u$ as axis because by (4),

$$F_p(u) = u \cos \theta + (u \times u) \sin \theta + (u, u) u (1 - \cos \theta)$$

$$= u \cos \theta + u (1 - \cos \theta)$$

$$= u. \quad (5)$$

Changing basis from $i, j, k$ to $u_1 = u, u_2, u_3$ which is orthonormal basis of right hand system, we get $F_p$ from (4) as,

$$\begin{aligned}
F_p(u_1) &= u_1 \\
F_p(u_2) &= u_2 \cos \theta + u_3 \sin \theta \\
F_p(u_3) &= -u_2 \sin \theta + u_3 \cos \theta
\end{aligned}$$

Hence

$$F_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix},$$

with respect to basis $u_1, u_2, u_3$. It follows that $F_p$ has $\theta$ as angle of rotation. Computing $F_p(i), F_p(j), F_p(k)$ with standard basis, we summarize as:
\textbf{Theorem 1} The rotation \( g(\theta; u) \in SO(3) \) of \( \mathbb{R}^3 = \text{ImH} \) with axis \( u \in \text{ImH} \), \(|u| = 1\) and angle \( \theta \), is given by

\[
g(\theta; u) = \text{Ad} \left( \exp \frac{\theta}{2} u \right)
= \begin{pmatrix}
(1 - \alpha^2) \cos \theta + \alpha^2 & ab - c \sin \theta - ab \cos \theta & ca + b \sin \theta - ca \cos \theta \\
ab + c \sin \theta - ab \cos \theta & (1 - b^2) \cos \theta + b^2 & bc - a \sin \theta - bc \cos \theta \\
ca - b \sin \theta - ca \cos \theta & bc + a \sin \theta - bc \cos \theta & (1 - c^2) \cos \theta + c^2
\end{pmatrix}.
\]

And every rotation \( g \in SO(3) \) can be written as the form: \( g = g(\theta; u) \) for some axis \( u \) and angle \( \theta \).

3 \ Product of Rotations

Let \( \rho = e^{\theta u/2} = \cos \frac{\theta}{2} + u \sin \frac{\theta}{2} \), \( \rho_1 = e^{\theta_1 u_1/2} = \cos \frac{\theta_1}{2} + u_1 \sin \frac{\theta_1}{2} \) and \( \rho_2 = e^{\theta_2 u_2/2} = \cos \frac{\theta_2}{2} + u_2 \sin \frac{\theta_2}{2} \). Consider the product of rotations:

\[
g(\theta; u) = g(\theta_2; u_2)g(\theta_1; u_1), \quad \text{i.e.,} \quad F_\rho = F_{\rho_2}F_{\rho_1} = F_{\rho_2\rho_1}.
\]

Then since kernel of \( \rho \mapsto F_\rho \) is \( \{ \pm 1 \} \),

\[
\rho = \varepsilon \rho_2\rho_1 \quad (\varepsilon = \pm 1).
\]

From the formula

\[
xy = -\langle x, y \rangle + x \times y, \quad x, y \in \text{ImH},
\]

we get

\[
\rho_2\rho_1 = \left( \cos \frac{\theta_2}{2} + u_2 \sin \frac{\theta_2}{2} \right) \left( \cos \frac{\theta_1}{2} + u_1 \sin \frac{\theta_1}{2} \right)
= \cos \frac{\theta_2}{2} \cos \frac{\theta_1}{2} + u_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_1}{2} + u_1 \cos \frac{\theta_2}{2} \sin \frac{\theta_1}{2} + u_2 u_1 \sin \frac{\theta_2}{2} \sin \frac{\theta_1}{2}
= \cos \frac{\theta_2}{2} \cos \frac{\theta_1}{2} - \langle u_2, u_1 \rangle \sin \frac{\theta_2}{2} \sin \frac{\theta_1}{2}
+ u_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_1}{2} + u_1 \cos \frac{\theta_2}{2} \sin \frac{\theta_1}{2} + (u_2 \times u_1) \sin \frac{\theta_2}{2} \sin \frac{\theta_1}{2}.
\]

Hence,

\[
\cos \frac{\theta}{2} + u \sin \frac{\theta}{2} = \varepsilon \left( \cos \frac{\theta_2}{2} \cos \frac{\theta_1}{2} - \langle u_2, u_1 \rangle \sin \frac{\theta_2}{2} \sin \frac{\theta_1}{2}
+ u_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_1}{2} + u_1 \cos \frac{\theta_2}{2} \sin \frac{\theta_1}{2} + (u_2 \times u_1) \sin \frac{\theta_2}{2} \sin \frac{\theta_1}{2} \right).
\]

Comparing real and imaginary parts we get the product formula:
The axis $u$ and angle $\theta$ of product rotation is determined by this formula. Consider the easy case $U_1 = U_2 = u'$. Then rotations in 3-space is in a plane. Since $(u' \cdot u', u' \times u') = 1$, $e = \cos^{-1} \frac{\theta}{2}$.

It is addition formula of sine and cosine.

4 Group Structure on $SO(3) \cong \mathbb{RP}^3$

We have several relations among $g(\theta; u)$'s:

$$g(0; u) = g(\theta; 0) = I,$$

$$g(\theta + 2\pi; u) = g(\theta; u), \quad g(\theta; u)^{-1} = g(-\theta; u) = g(\theta; -u),$$

for any $u \in \mathbb{R}^3$, $|u| = 1$, $\theta \in \mathbb{R}$ and hence,

$$g(\theta + \pi; u) = g(\theta - \pi; u) = g(\pi - \theta; u).$$

Therefore we can strengthen theorem 1 in part: every rotation $g \in SO(3)$ is of the form: $g = g(\theta; u)$ with $0 \leq \theta \leq \pi$. For any $v \in \text{ImH}$, $v \neq 0$, let $v = \theta u$, $\theta = |v|$, $u = v/|v|$ be its polar decomposition. Define $g(v) \in SO(3)$ by

$$g(v) = g(\theta; u) = \text{Ad} \left( \exp \frac{v}{2} \right)$$

and call $v \in \text{ImH}$ the axis vector of $g(v) \in SO(3)$. An axis vector indicates the axis and angle of a rotation by its direction and length. We then have a surjection

$$g : \text{ImH} \xrightarrow{\exp} S^3 \xrightarrow{F} SO(3).$$

We know $g(D^3) = SO(3)$ where $D^3 = \{ v \in \text{ImH} \mid |v| \leq \pi \}$. Since $g(\pi; u) = g(\pi; -u)$, $g|D^3$ induces a homeomorphism of topological spaces:

$$g : D^3/(v \sim -v, |v| = \pi) \cong S^3/(x \sim -x) \cong SO(3).$$

$\mathbb{RP}^3 = S^3/(x \sim -x)$ is the 3-dimensional real projective space. Since $D^3/(v \sim -v, |v| = \pi) = \text{ImH}/\sim$ where $v \sim w \iff g(v) = g(w)$, we here look on $\mathbb{RP}^3$ as
the set of all the axes vectors modulo some equivalence. The rotation group \( SO(3) \) induces a group structure on this \( \mathbb{R}P^3 \) as:

**Theorem 2** Let \( \mathbb{R}P^3 = D^3/(v \sim -v, |v| = \pi) \) = the set of all the axes vectors of rotations modulo equivalence. Then the above \( g \) induces a group structure on \( \mathbb{R}P^3 = SO(3) \). In the group,

1. the unit element is zero vector.
2. the inverse of \( v \) is \(-v\).
3. the product of 2 axes vectors is computed by the product formula \((7)\) modulo equivalence.

## 5 Proof of Formulas

We give proofs of some facts and formulas. Refer to [2].

The exponential map \( \exp : \text{Im} \mathbb{H} \to S^3 \) is surjective.

**Proof.** Let \( \rho = a + bu \in S^3 \), \( a, b \in \mathbb{R}, u \in \text{Im} \mathbb{H}, |u| = 1 \). From \( |\rho|^2 = a^2 + b^2 = 1 \), we get \( a = \cos \theta, b = \sin \theta \) for some \( \theta \). So \( \exp(\theta u) = e^{\theta u} = \cos \theta + u \sin \theta = \rho. \)

\[
(3) \quad xy = -(x, y) + x \times y, \quad (6) \quad \langle x, y \rangle = -\frac{1}{2}(xy + yx)
\]

**Proof.** Let \( x = x_1i + x_2j + x_3k, y = y_1i + y_2j + y_3k \in \text{Im} \mathbb{H}, \)

\[
x y = (x_1i + x_2j + x_3k)(y_1i + y_2j + y_3k) = -(x_1y_1 + x_2y_2 + x_3y_3) + (x_2y_3 - y_2x_3)i + (x_3y_1 - y_3x_1)j + (x_1y_2 - y_1x_2)k
\]

For

\[
\langle x, y \rangle = x_1y_1 + x_2y_2 + x_3y_3,
\]

\[
x \times y = \begin{vmatrix} x_2 & y_2 & i \\ x_3 & y_3 & j \\ x_1 & y_1 & k \end{vmatrix} + \begin{vmatrix} x_3 & y_3 & j \\ x_1 & y_1 & k \\ x_2 & y_2 & i \end{vmatrix}
\]

we have

\[
xy = -(x, y) + x \times y.
\]

It follows that immediately,

\[
\langle x, y \rangle = -\frac{1}{2}(xy + yx)
\]

\[
x \times y = \frac{1}{2}(xy - yx).
\]
This completes the proof. □

\[(4) \quad F_\rho(x) = x \cos \theta + (u \times x) \sin \theta + (1 - \cos \theta) \langle u, x \rangle u \]

**Proof.** Let \( \rho = e^{\theta u/2} = \cos \frac{\theta}{2} + u \sin \frac{\theta}{2} \in Sp(1), \ x \in \text{Im}\mathbb{H}. \)

\[
F_\rho(x) = \rho x \rho^{-1} = \left( \cos \frac{\theta}{2} + u \sin \frac{\theta}{2} \right) x \left( \cos \frac{\theta}{2} - u \sin \frac{\theta}{2} \right)
\]
\[
= x \cos^2 \frac{\theta}{2} - u xu \sin^2 \frac{\theta}{2} + u x \sin \frac{\theta}{2} \cos \frac{\theta}{2} - xu \sin \frac{\theta}{2} \cos \frac{\theta}{2}
\]
\[
= x \cos^2 \frac{\theta}{2} - u xu \sin^2 \frac{\theta}{2} + (u \times x) 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}.
\]

Here \( u xu = x - 2 \langle u, x \rangle \) because by (6),
\[
u xu = \{-\langle u, x \rangle + (u \times x)\}u = -\langle u, x \rangle u + (u \times x)u.
\]

And by (8),
\[
u xu = -\langle u, x \rangle u + \frac{1}{2}(u xu + x) \Rightarrow xu = x - 2 \langle u, x \rangle u.
\]

Therefore
\[
F_\rho(x) = x \cos^2 \frac{\theta}{2} + (2 \langle u, x \rangle u - x) \sin^2 \frac{\theta}{2} + (u \times x) \sin \theta
\]
\[
= x \cos \theta + (u \times x) \sin \theta - 2 \sin^2 \frac{\theta}{2} \langle u, x \rangle u
\]
\[
= x \cos \theta + (u \times x) \sin \theta + (1 - \cos \theta) \langle u, x \rangle u.
\]

This completes the proof. □

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**References**

