On Rotation Matrices of given Axes and Angles and the Group Structure on $SO(3)$

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(Received October 31, 2003)

Abstract

We treat rotation matrices of given axes and angles in the space $\mathbb{R}^3 = \text{Im}\mathbb{H}$ of pure imaginary quaternions. We give a product formula of rotation matrices of given axes vectors and so explain the group structure on $SO(3) \simeq \mathbb{R}P^3$ from the viewpoint of axes and angles.

1 Introduction

We give the matrix expression $g(\theta; u) \in SO(3)$ of rotation in $\mathbb{R}^3$ of given axis $u \in \mathbb{R}^3$, $|u| = 1$ and angle $\theta$ by using the adjoint representation $\text{Ad}: S^3 = Sp(1) \rightarrow SO(3)$, as the following form:

$$g(\theta; u) = g(\theta u) = \text{Ad} \left( \exp \frac{\theta u}{2} \right)$$

where $u \in \mathbb{R}^3$ is identified with a quaternion in $\text{Im}\mathbb{H}$ and $\theta u \in \mathbb{R}^3$ is called the axis vector of the rotation. $g(\theta; u)$ is to rotate clockwise around the axis $u$ with angle $\theta$. The description is classically known as the Cayley-Klein parameter, and is equivalent to that given by the adjoint representation of $SU(2)$. We next give the product formula:

$$g(\theta_1; u_1)g(\theta_2; u_2) = g(\theta_3; u_3)$$

and so look closely at the group structure in $SO(3) = \mathbb{R}P^3$ which is a closed ball of radius $\pi$ in $\mathbb{R}^3$ whose antipodal points in the boundary are identified.
2 Description of Rotational Transformation by Quaternions

We identify the set $\text{Im}\mathbb{H}$ of all pure imaginary quaternions with the real 3-dimensional space $\mathbb{R}^3$ by a linear isomorphism over $\mathbb{R}$:

\[
\begin{pmatrix}
\mathbb{R}^3 \\
\psi
\end{pmatrix} \sim \begin{pmatrix}
\text{Im}\mathbb{H} \\
\psi
\end{pmatrix}
\]

\[
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} \mapsto ai + bj + ck
\]  

(1)

Let $x = x_1i + x_2j + x_3k$, $y = y_1i + y_2j + y_3k \in \text{Im}\mathbb{H}$. Define an inner product in $\text{Im}\mathbb{H}$ by

\[
\langle x, y \rangle = x_1y_1 + x_2y_2 + x_3y_3.
\]

Then identification (1) is an isomorphism of Euclidean spaces.

Let $S^3 = Sp(1) = \{\rho \in \mathbb{H} | |\rho| = 1\}$. For $\rho \in S^3$, we denote the adjoint representation of $S^3 = Sp(1)$ by $F_\rho$:

\[
F_\rho = \text{Ad}_\rho : x \mapsto \rho x \rho^{-1}, \quad \text{Im}\mathbb{H} \to \text{Im}\mathbb{H}.
\]

(2)

For any $u \in \text{Im}\mathbb{H}$, $|u| = 1$, we have $u^2 = -1$. Hence the exponential is given by

\[
e^{\theta u} = \cos \theta + u \sin \theta, \quad \theta \in \mathbb{R}.
\]

The exponential map $\exp : \text{Im}\mathbb{H} \to S^3$ is then surjective. We show that

1. The sequence: $1 \to \{\pm 1\} \to S^3 \xrightarrow{F} SO(3) \to 1$ is exact,

2. If $\rho = e^{\theta u}(u \in \text{Im}\mathbb{H}, |u| = 1)$ then $F_\rho$ has $u$ as axis and $\theta$ as angle.

2.1 $F(S^3) = SO(3)$ and $\text{Ker } F = \{\pm 1\}$

$\text{Ker } F = \{\pm 1\}$ is a consequence of center($\mathbb{H}$)=$\mathbb{R}$ because $\mathbb{R} \cap S^3 = \{\pm 1\}$. The formula

\[
\langle x, y \rangle = -\frac{1}{2}(xy + yx)
\]

(3)

shows not changing an inner product by $F_\rho$, i.e.,

\[
\langle F_\rho(x), F_\rho(y) \rangle = \langle x, y \rangle.
\]

So $F(S^3) \subset O(3)$. The map $\rho \mapsto \det F_\rho$ is a continuous map from a connected $S^3$ to $\{\pm 1\}$, we have $\det F_\rho = +1$ and so $F(S^3) \subset SO(3)$. Since $\dim S^3 = \dim SO(3) = 3$ and $F$ is a continuous homomorphism between connected groups with discrete kernel, we know that $F(S^3) = SO(3)$. 
2.2 Axes and Angles

We show that $F_\rho (\rho = e^{\theta u/2})$ has $u$ as axis and $\theta$ as clockwise angle of rotation. We use the formula

$$F_\rho (x) = x \cos \theta + (u \times x) \sin \theta + \langle u, x \rangle u (1 - \cos \theta) \quad (4)$$

where $u \times x$ is an outer product given by

$$x \times y = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} i + \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} j + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k. \quad (5)$$

$F_\rho$ has $u$ as axis because by (4),

$$F_\rho (u) = u \cos \theta + (u \times u) \sin \theta + \langle u, u \rangle u (1 - \cos \theta)$$

$$= u \cos \theta + u (1 - \cos \theta)$$

$$= u.$$

Changing basis from $i, j, k$ to $u_1 = u, u_2, u_3$ which is orthonormal basis of right hand system, we get $F_\rho$ from (4) as,

$$\begin{cases} F_\rho (u_1) = u_1 \\ F_\rho (u_2) = u_2 \cos \theta + u_3 \sin \theta \\ F_\rho (u_3) = -u_2 \sin \theta + u_3 \cos \theta \end{cases}$$

Hence

$$F_\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix},$$

with respect to basis $u_1, u_2, u_3$. It follows that $F_\rho$ has $\theta$ as angle of rotation. Computing $F_\rho (i), F_\rho (j), F_\rho (k)$ with standard basis, we summarize as:
**Theorem 1** The rotation \( g(\theta; u) \in SO(3) \) of \( \mathbb{R}^3 = \mathbb{H} \) with axis \( u \in \mathbb{H} \), \(|u| = 1\) and angle \( \theta \), is given by

\[
g(\theta; u) = \text{Ad} \left( \exp \frac{\theta}{2} u \right)
\]

\[
= \begin{pmatrix}
(1 - a^2) \cos \theta + a^2 & ab - c \sin \theta - ab \cos \theta & ca + b \sin \theta - ca \cos \theta \\
ab + c \sin \theta - ab \cos \theta & (1 - b^2) \cos \theta + b^2 & bc - a \sin \theta - bc \cos \theta \\
ca - b \sin \theta - ca \cos \theta & bc + a \sin \theta - bc \cos \theta & (1 - c^2) \cos \theta + c^2
\end{pmatrix}.
\]

And every rotation \( g \in SO(3) \) can be written as the form: \( g = g(\theta; u) \) for some axis \( u \) and angle \( \theta \).

### 3 Product of Rotations

Let \( \rho = e^{\theta u/2} = \cos \frac{\theta}{2} + u \sin \frac{\theta}{2}, \rho_1 = e^{\theta_1 u_1/2} = \cos \frac{\theta_1}{2} + u_1 \sin \frac{\theta_1}{2} \) and \( \rho_2 = e^{\theta_2 u_2/2} = \cos \frac{\theta_2}{2} + u_2 \sin \frac{\theta_2}{2} \). Consider the product of rotations:

\[
g(\theta; u) = g(\theta_2; u_2)g(\theta_1; u_1), \quad \text{i.e.,} \quad F_\rho = F_{\rho_2}F_{\rho_1} = F_{\rho_2\rho_1},
\]

Then since kernel of \( \rho \mapsto F_\rho \) is \( \{\pm 1\} \),

\[
\rho = \varepsilon \rho_2\rho_1 \quad (\varepsilon = \pm 1).
\]

From the formula

\[
xy = -\langle x, y \rangle + x \times y, \quad x, y \in \mathbb{H},
\]

we get

\[
\rho_2\rho_1 = \begin{pmatrix}
\cos \frac{\theta_2}{2} + u_2 \sin \frac{\theta_2}{2} \\
\cos \frac{\theta_1}{2} + u_1 \sin \frac{\theta_1}{2}
\end{pmatrix} \begin{pmatrix}
\cos \frac{\theta_1}{2} + u_1 \sin \frac{\theta_1}{2} \\
\cos \frac{\theta_2}{2} + u_2 \sin \frac{\theta_2}{2}
\end{pmatrix}
\]

\[
= \cos \frac{\theta_2}{2} \cos \frac{\theta_1}{2} + u_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_1}{2} + u_1 \cos \frac{\theta_2}{2} \sin \frac{\theta_1}{2} + u_2 u_1 \sin \frac{\theta_2}{2} \sin \frac{\theta_1}{2}
\]

\[
= \cos \frac{\theta_2}{2} \cos \frac{\theta_1}{2} - \langle u_2, u_1 \rangle \sin \frac{\theta_2}{2} \sin \frac{\theta_1}{2}
\]

\[
+ u_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_1}{2} + u_1 \cos \frac{\theta_2}{2} \sin \frac{\theta_1}{2} + (u_2 \times u_1) \sin \frac{\theta_2}{2} \sin \frac{\theta_1}{2}.
\]

Hence,

\[
\cos \frac{\theta}{2} + u \sin \frac{\theta}{2} = \varepsilon \left\{ \cos \frac{\theta_2}{2} \cos \frac{\theta_1}{2} - \langle u_2, u_1 \rangle \sin \frac{\theta_2}{2} \sin \frac{\theta_1}{2}
\]

\[
+ u_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_1}{2} + u_1 \cos \frac{\theta_2}{2} \sin \frac{\theta_1}{2} + (u_2 \times u_1) \sin \frac{\theta_2}{2} \sin \frac{\theta_1}{2} \right\}.
\]

Comparing real and imaginary parts we get the product formula:
The axis \( u \) and angle \( \theta \) of product rotation is determined by this formula.

Consider the easy case \( u_1 = u_2 = u' \). Then rotations in 3-space is in a plane. Since 
\[
\begin{align*}
\cos \frac{\theta}{2} &= \varepsilon \left( \cos \frac{\theta_2}{2} \cos \frac{\theta_1}{2} - \langle u_2, u_1 \rangle \sin \frac{\theta_2}{2} \sin \frac{\theta_1}{2} \right) \\
u \sin \frac{\theta}{2} &= \varepsilon \left( u_2 \sin \frac{\theta_2}{2} \cos \frac{\theta_1}{2} + u_1 \cos \frac{\theta_2}{2} \sin \frac{\theta_1}{2} + (u_2 \times u_1) \sin \frac{\theta_2}{2} \sin \frac{\theta_1}{2} \right)
\end{align*}
\]

It is addition formula of sine and cosine.

4 Group Structure on \( SO(3) \simeq \mathbb{R}P^3 \)

We have several relations among \( g(\theta; u) \)'s:

\[
g(0; u) = g(\theta; 0) = I,
\]

\[
g(\theta + 2\pi; u) = g(\theta; u), \quad g(\theta; u)^{-1} = g(-\theta; u) = g(\theta; -u),
\]

for any \( u \in \mathbb{R}^3, \ |u| = 1, \ \theta \in \mathbb{R} \) and hence,

\[
g(\theta + \pi; u) = g(\theta - \pi; u) = g(\pi - \theta; -u).
\]

Therefore we can strengthen theorem 1 in part: every rotation \( g \in SO(3) \) is of the form: \( g = g(\theta; u) \) with \( 0 \leq \theta \leq \pi \). For any \( v \in \text{Im} \mathbb{H}, \ v \neq 0 \), let \( v = \theta u, \ \theta = |v|, \ u = v/|v| \) be its polar decomposition. Define \( g(v) \in SO(3) \) by

\[
g(v) = g(\theta; u) = \text{Ad} \left( \exp \frac{v}{2} \right)
\]

and call \( v \in \text{Im} \mathbb{H} \) the axis vector of \( g(v) \in SO(3) \). An axis vector indicates the axis and angle of a rotation by its direction and length. We then have a surjection

\[
g : \text{Im} \mathbb{H} \xrightarrow{\exp} S^3 \xrightarrow{E} SO(3).
\]

We know \( g(D^3) = SO(3) \) where \( D^3 = \{ v \in \text{Im} \mathbb{H}, \ |v| \leq \pi \} \). Since \( g(\pi; u) = g(\pi; -u) \),

\( g|D^3 \) induces a homeomorphism of topological spaces:

\[
g : D^3/(v \sim -v, \ |v| = \pi) \xrightarrow{\sim} S^3/(x \sim -x) \xrightarrow{\sim} SO(3).
\]

\( \mathbb{R}P^3 = S^3/(x \sim -x) \) is the 3-dimensional real projective space. Since \( D^3/(v \sim -v, \ |v| = \pi) = \text{Im} \mathbb{H}/\sim \) where \( v \sim w \iff g(v) = g(w) \), we here look on \( \mathbb{R}P^3 \) as
the set of all the axes vectors modulo some equivalence. The rotation group \( SO(3) \) induces a group structure on this \( \mathbb{RP}^3 \) as:

**Theorem 2** Let \( \mathbb{RP}^3 = D^3/(v \sim -v, \ |v| = \pi) \) = the set of all the axes vectors of rotations modulo equivalence. Then the above \( g \) induces a group structure on \( \mathbb{RP}^3 = SO(3) \). In the group,

1. the unit element is zero vector.
2. the inverse of \( v \) is \( -v \).
3. the product of 2 axes vectors is computed by the product formula (7) modulo equivalence.

## 5 Proof of Formulas

We give proofs of some facts and formulas. Refer to [2].

The exponential map \( \exp : \text{Im} \mathbb{H} \to S^3 \) is surjective.

**Proof.** Let \( \rho = a + bu \in S^3, a, b \in \mathbb{R}, u \in \text{Im} \mathbb{H}, \ |u| = 1 \). From \( |\rho|^2 = a^2 + b^2 = 1 \), we get \( a = \cos \theta, b = \sin \theta \) for some \( \theta \). So \( \exp(\theta u) = e^{\theta u} = \cos \theta + u \sin \theta = \rho \). \( \Box \)

\[
(3) \quad xy = -\langle x, y \rangle + x \times y, \quad (6) \quad \langle x, y \rangle = -\frac{1}{2}(xy + yx)
\]

**Proof.** Let \( x = x_1 i + x_2 j + x_3 k, y = y_1 i + y_2 j + y_3 k \in \text{Im} \mathbb{H}, \)

\[
xy = (x_1 i + x_2 j + x_3 k)(y_1 i + y_2 j + y_3 k)
= -(x_1 y_1 + x_2 y_2 + x_3 y_3) + (x_2 y_3 - y_3 x_2)i + (x_3 y_1 - y_3 x_3)j + (x_1 y_2 - y_2 x_1)k
= -(x_1 y_1 + x_2 y_2 + x_3 y_3) + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} i + \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} j + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k.
\]

For

\[
\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3,
\]

\[
x \times y = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} i + \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} j + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k,
\]

we have

\[
xy = -\langle x, y \rangle + x \times y.
\]

It follows that immediately,

\[
\langle x, y \rangle = -\frac{1}{2}(xy + yx)
\]

\[
x \times y = \frac{1}{2}(xy - yx).
\]
This completes the proof. □

\( F_\rho(x) = x \cos \theta + (u \times x) \sin \theta + (1 - \cos \theta) \langle u, x \rangle u \)

**Proof.** Let \( \rho = e^{i \theta/2} = \cos \frac{\theta}{2} + u \sin \frac{\theta}{2} \in Sp(1), x \in \text{ImH}. \)

\[
F_\rho(x) = \rho x \rho^{-1} = \left( \cos \frac{\theta}{2} + u \sin \frac{\theta}{2} \right) x \left( \cos \frac{\theta}{2} - u \sin \frac{\theta}{2} \right)
\]

\[
= x \cos^2 \frac{\theta}{2} - u x u \sin^2 \frac{\theta}{2} + u x \sin \frac{\theta}{2} \cos \frac{\theta}{2} - x u \sin \frac{\theta}{2} \cos \frac{\theta}{2}
\]

\[
= x \cos^2 \frac{\theta}{2} - u x u \sin^2 \frac{\theta}{2} + (u \times x) 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}.
\]

Here \( ux = x - 2\langle u, x \rangle \) because by (6),

\( uxu = \langle -u, x \rangle + (u \times x) \rangle u = -\langle u, x \rangle u + (u \times x) u. \)

And by (8),

\( u x u = -\langle u, x \rangle u + \frac{1}{2} (u x + x) \Rightarrow u x u = x - 2\langle u, x \rangle u. \)

Therefore

\[
F_\rho(x) = x \cos^2 \frac{\theta}{2} + (2\langle u, x \rangle u - x) \sin^2 \frac{\theta}{2} + (u \times x) \sin \theta
\]

\[
= x \cos \theta + (u \times x) \sin \theta - 2 \sin^2 \frac{\theta}{2} \langle u, x \rangle u
\]

\[
= x \cos \theta + (u \times x) \sin \theta + (1 - \cos \theta) \langle u, x \rangle u.
\]

This completes the proof. □

**Acknowledgment.** The authors thanks T. Sugawara for many helpful discussions and advices.

**References**

