



Title	Some Holder estimates for on convex domains in C^3 with real analytic boundary
Author(s)	Adachi, Kenzo; Cho, Hong Rae
Citation	長崎大学教育学部自然科学研究報告. vol.58, p.1-7; 1998
Issue Date	1998-03-25
URL	http://hdl.handle.net/10069/6177
Right	

This document is downloaded at: 2020-10-26T19:27:43Z

Some Hölder estimates for $\bar{\partial}$ on convex domains in \mathbf{C}^3 with real analytic boundary

Kenzō ADACHI and Hong Rae CHO*

Department of Mathematics, Faculty of Education,
Nagasaki University, Nagasaki 852, Japan

(Received Oct. 31, 1997)

abstract

Let $D \subset\subset \mathbf{C}^3$ be a convex domain with real analytic boundary. We get some Hölder estimates for $\bar{\partial}$ on the domain D .

1. Introduction and statement of the result.

Hölder estimates for $\bar{\partial}u = f$ on strongly pseudoconvex domains were obtained by Kerzman[3] and Henkin and Romanov[2]. Range[5] obtained Hölder estimates for $\bar{\partial}$ on convex domains in \mathbf{C}^2 with real analytic boundary. Also, there are several results about Hölder estimates for $\bar{\partial}$ in the case of dimension 2 ([8],[9]). In the higher dimensional case, Polking[4] studied the $\bar{\partial}$ problem in convex domains. Bruna and Castillo[1] obtained Hölder estimates for $\bar{\partial}$ in some convex domains in \mathbf{C}^n with real analytic boundary which contain complex ellipsoids. However, it is an open problem to obtain Hölder estimates for $\bar{\partial}$ on convex domains in $\mathbf{C}^n (n \geq 3)$ with real analytic boundary. In this paper, we obtain some Hölder estimates for $\bar{\partial}$ on convex domains in \mathbf{C}^3 with real analytic boundary.

Let $D \subset\subset \mathbf{C}^n$ be a convex domain with C^2 defining function r . We define $\phi(\zeta, z) = \langle \bar{\partial}r(\zeta), \zeta - z \rangle$. Let $T : C_{0,1}(\bar{D}) \rightarrow C(D)$ be the Henkin solution operator for $\bar{\partial}$ and assume that f is a $\bar{\partial}$ -closed $(0, 1)$ form. One has

$$T(f) = H(f) + K(f),$$

where

(1.1)

$$H(f)(z) = \sum_{k=0}^{n-2} C_k \int_{bD} f(\zeta) \wedge \frac{\partial_{\zeta} |\zeta - z|^2 \wedge \bar{\partial}r(\zeta) \wedge (\bar{\partial}\bar{\partial}r(\zeta))^k \wedge (\bar{\partial}\bar{\partial}|\zeta - z|^2)^{n-k-2}}{\phi(\zeta, z)^{k+1} |\zeta - z|^{2(n-k-1)}}$$

*Department of Mathematics Education, Andong National University, Andong Kyungbuk 760-749, Korea

and $K(f)$ is given by integrating f against the Bochner-Martinelli kernel over D . We denote by $\Lambda_\alpha(D)$ the classical Lipschitz space of order α . It is well known that $|K(f)|_{\Lambda_\alpha} \leq C_\alpha |f|_{L^\infty}$ for $0 < \alpha < 1$, so that it is enough to consider $H(f)$.

Let $D \subset\subset \mathbf{C}^n$ be a convex domain with real analytic boundary. By Range[6], there are a positive number $\epsilon > 0$ and a positive even integer $m \geq 2$ such that

$$(1.2) \quad |\phi(\zeta, z)| \gtrsim |\operatorname{Im} \phi(\zeta, z)| + \operatorname{dist}(z, bD) + |\zeta - z|^m$$

for $(\zeta, z) \in bD \times \bar{D}$ with $|\zeta - z| < \epsilon$. If $m = 2$, then by Henkin and Romanov[2], for $f \in L^\infty(D)$ we have $Tf \in \Lambda_{\frac{1}{2}}(D)$. Hence we assume that $m \geq 4$. Now we state our main result.

THEOREM. *Let $D \subset\subset \mathbf{C}^3$ be a convex domain with real analytic boundary. Let $T : C_{0,1}(\bar{D}) \rightarrow C(D)$ be the Henkin solution operator for $\bar{\partial}$ on D . For $1 - \frac{3}{m} < \alpha < 1$ if $f \in \Lambda_\alpha(D)$, then $Tf \in \Lambda_\beta(D)$, where $\beta = \alpha - 1 + \frac{3}{m}$.*

Remark. In particular, the result gives a useful sufficient condition for the continuity on \bar{D} of the solution $u = Tf$ for the equation $\bar{\partial}u = f$.

2. Some lemmas.

LEMMA 1. *For $\beta > 2, \delta > 0$, and a positive integer $m \geq 4$,*

$$\int_{\substack{x \in \mathbf{R}^4 \\ |x| \leq 1}} \frac{dx}{(\delta + |x|^m)^\beta |x|} \lesssim \delta^{-\beta + \frac{3}{m}}.$$

PROOF. By using the polar coordinates, we get

$$\begin{aligned} \int_{\substack{x \in \mathbf{R}^4 \\ |x| \leq 1}} \frac{dx}{(\delta + |x|^m)^\beta |x|} &\lesssim \int_0^1 \frac{\rho^2 d\rho}{(\delta + \rho^m)^\beta} \\ &\lesssim \delta^{-\beta + \frac{3}{m}} \int_0^\infty \frac{s^2 ds}{(1 + s^m)^\beta} \\ &\lesssim \delta^{-\beta + \frac{3}{m}}. \end{aligned}$$

The following lemma is well known(cf. Range[7]):

LEMMA 2. *Let $D \subset\subset \mathbf{C}^n$ be a bounded domain with C^2 defining function r . Suppose that $f \in C^1(D)$ and that for some $0 < \alpha < 1$ there is a constant C_f , such that*

$$|df(z)| \leq C_f |r(z)|^{\alpha-1}, \quad z \in D.$$

Then $f \in \Lambda_\alpha(D)$.

We now show that $\text{Im } \phi(\zeta, z)$ can be used as a local coordinate on bD . This can be proved by Range ([7], V-3). But for the reader's convenience, we give the proof.

LEMMA 3. *There are positive constants M, a , and $\eta \leq \epsilon$, and, for each z with $\text{dist}(z, bD) \leq a$, there is a C^∞ local coordinate system $(t_1, \dots, t_6) = t = t(\zeta, z)$ on $B(z, \eta)$ such that the following hold:*

$$\begin{aligned} t_1(\zeta, z) &= r(\zeta), \quad t(z, z) = (r(z), 0, \dots, 0) \quad \text{and} \quad t_2(\zeta, z) = \text{Im } \phi(\zeta, z), \\ |t(\zeta, z)| &< 1 \quad \text{for } \zeta \in B(z, \eta), \\ |J_{\mathbf{R}}(t(\cdot, z))| &\leq M, \quad |\det J_{\mathbf{R}}(t(\cdot, z))| \geq \frac{1}{M}. \end{aligned}$$

PROOF. Fix $z \in bD$. Since

$$\phi(\zeta, z) = \sum_{j=1}^3 \frac{\partial r}{\partial \zeta_j}(\zeta)(\zeta_j - z_j),$$

it follows that

$$(2.1) \quad d_\zeta \phi(z, z) = \partial_\zeta r(z).$$

Thus, at the point $\zeta = z$, one obtains

$$\begin{aligned} d_\zeta \text{Im } \phi \wedge d_\zeta r &= \frac{1}{2i} (\partial_\zeta r - \bar{\partial}_\zeta r) \wedge (\partial_\zeta r + \bar{\partial}_\zeta r) \\ &= \frac{1}{i} \partial r \wedge \bar{\partial} r \neq 0. \end{aligned}$$

Hence, we have the result.

3. Proof of the theorem.

By differentiating under the integral in (1.1) one obtains

$$d_z H(f)(z) = I_1 f(z) + I_2 f(z) + I_3 f(z) + I_4 f(z),$$

where

$$\begin{aligned} I_1 f(z) &= \int_{bD} f(\zeta) \wedge \frac{A_1(\zeta, z)}{\phi(\zeta, z) |\zeta - z|^4}, \\ I_2 f(z) &= \int_{bD} f(\zeta) \wedge \frac{A_2(\zeta, z)}{\phi(\zeta, z)^2 |\zeta - z|^2}, \\ I_3 f(z) &= \int_{bD} f(\zeta) \wedge \frac{A_3(\zeta, z)}{\phi(\zeta, z)^2 |\zeta - z|^3}, \\ I_4 f(z) &= \int_{bD} f(\zeta) \wedge \frac{A_4(\zeta, z)}{\phi(\zeta, z)^3 |\zeta - z|}. \end{aligned}$$

The expressions $A_j(\zeta, z)$ are smooth double forms on $\overline{D} \times \overline{D}$, of degree 1 in z and type (3,1) in ζ . Let $\gamma > 0$ be the smaller of the constants a and η in Lemma 3. By using Lemma 2, the compactness of bD , and a partition of unity, the estimation of $I_j f(z)$ is reduced to proving the estimates

$$|I_j(\chi f)(z)| \leq C_\alpha |f|_{\Lambda_\alpha} |r(z)|^{\alpha + \frac{3}{m} - 2} \quad \text{for } z \in D,$$

where χ is a compactly supported cut off function in $B(z, \gamma)$. We have

$$\begin{aligned} |I_1(\chi f)(z)| &\lesssim |f|_{L^\infty} \int_{bD \cap B(z, \gamma)} \frac{d\sigma(\zeta)}{|\phi(\zeta, z)| |\zeta - z|^4} \\ &\lesssim |f|_{L^\infty} \int_{|t'| \leq 1} \frac{dt'}{(|t_2| + |r(z)|) |t'|^4} \\ &\lesssim C_\beta |f|_{L^\infty} |r(z)|^{-\beta}, \quad 0 < \beta < 1. \end{aligned}$$

We decompose

$$(3.1) \quad f(t) = [f(t_1, t_2, t_3, t_4, t_5, t_6) - f(0, 0, t_3, t_4, t_5, t_6)] + f(0, 0, t_3, t_4, t_5, t_6).$$

Corresponding to (3.1), we have $I_2(\chi f)(z) = I_{21}(z) + I_{22}(z)$ with $t' = (t_2, t_3, t_4, t_5, t_6)$,

$$\begin{aligned} |I_{21}(z)| &\lesssim |f|_{\Lambda_\alpha(bD)} \int_{|t'| \leq 1} \frac{dt'}{|\phi|^{2-\alpha} |t'|^2} \\ &\lesssim |f|_{\Lambda_\alpha(bD)} |r(z)|^{\alpha-1} \end{aligned}$$

and

$$(3.2) \quad I_{22}(z) = \int_{|t'| \leq 1} \frac{f(0, 0, t_3, t_4, t_5, t_6) \chi(t') A_2(t', z)}{\phi^2 |t'|^2} dt'.$$

In order to estimate (3.2), we first integrate by parts in t_2 . From (2.1) one also obtain that $2 d_\zeta \operatorname{Re} \phi = d_\zeta r$ at $\zeta = z$, therefore we have

$$d_\zeta \operatorname{Re} \phi \wedge d_\zeta \operatorname{Im} \phi \neq 0.$$

If we choose coordinates $t_1 = r(\zeta)$ and $t_2 = \operatorname{Im} \phi(\zeta, z)$ as in Lemma 3, then we have

$$\frac{\partial \operatorname{Re} \phi}{\partial t_2} = 0 \quad \text{and} \quad \frac{\partial \operatorname{Im} \phi}{\partial t_2} = 1.$$

Thus, it follows that

$$\frac{\partial \phi}{\partial t_2} = i.$$

Therefore we have

$$\phi^{-2} = i \frac{\partial}{\partial t_2} (\phi^{-1}).$$

Since in (3.2) the term involving f is independent of t_2 , and since χ has compact support in $B(z, \gamma)$, by integration by parts, one obtains

$$I_{22}(z) = -i \int_{|t'| \leq 1} \frac{f(0, 0, t_3, t_4, t_5, t_6)}{\phi} \frac{\partial}{\partial t_2} \left[\frac{\chi(t') A_2(t', z)}{|t'|^2} \right] dt',$$

which leads to

$$\begin{aligned} |I_{22}(z)| &\lesssim |f|_{L^\infty} \int_{|t'| \leq 1} \frac{dt'}{(|t_2| + |r(z)|) |t'|^3} \\ &\lesssim C_\beta |f|_{L^\infty} |r(z)|^{-\beta}, \quad 0 < \beta < 1. \end{aligned}$$

As in the case of $I_2(\chi f)$, we have $I_3(\chi f)(z) = I_{31}(z) + I_{32}(z)$, where

$$\begin{aligned} |I_{31}(z)| &\lesssim |f|_{\Lambda_\alpha(bD)} \int_{|t'| \leq 1} \frac{dt'}{|\phi|^{2-\alpha} |t'|^3} \\ &\lesssim |f|_{\Lambda_\alpha(bD)} |r(z)|^{\alpha-1} \end{aligned}$$

and

$$I_{32}(z) = \int_{|t'| \leq 1} \frac{f(0, 0, t_3, t_4, t_5, t_6) \chi(t') A_3(t', z)}{\phi^2 |t'|^3} dt'.$$

For $I_{32}(z)$, it follows that

$$|I_{32}(z)| \lesssim C_\beta |f|_{L^\infty} |r(z)|^{-\beta}, \quad 0 < \beta < 1.$$

For $I_4(\chi f)$, we have $I_4(\chi f)(z) = I_{41}(z) + I_{42}(z)$, where

$$|I_{41}(z)| \lesssim |f|_{\Lambda_\alpha(bD)} \int_{|t'| \leq 1} \frac{dt'}{|\phi|^{3-\alpha} |t'|}$$

and

$$I_{42}(z) = \int_{|t'| \leq 1} \frac{f(0, 0, t_3, t_4, t_5, t_6) \chi(t') A_4(t', z)}{\phi^3 |t'|} dt'.$$

By using Lemma 1, it follows that

$$\begin{aligned} \int_{|t'| \leq 1} \frac{dt'}{|\phi|^{3-\alpha} |t'|} &\lesssim \int_0^1 \int_{t''=(t_3, t_4, t_5, t_6)}^{|t''| \leq 1} \frac{dt'' dt_1}{(|t_2| + |r(z)| + |t''|^m)^{3-\alpha} |t''|} \\ &\lesssim \int_0^1 \frac{dt_2}{(|t_2| + |r(z)|)^{3-\alpha-\frac{3}{m}}} \\ &\lesssim |r(z)|^{\alpha+\frac{3}{m}-2}. \end{aligned}$$

Hence we have

$$|I_{41}| \lesssim |f|_{\Lambda_\alpha(bD)} |r(z)|^{\alpha + \frac{3}{m} - 2}.$$

Taking account of the relation

$$\phi^{-3} = -\frac{1}{2} \frac{\partial^2}{\partial t_2^2} (\phi^{-1}),$$

it follows that

$$I_{42}(z) = -\frac{1}{2} \int_{|t'| \leq 1} \frac{f(0, 0, t_3, t_4, t_5, t_6)}{\phi} \frac{\partial^2}{\partial t_2^2} \left[\frac{\chi(t') A_4(t', z)}{|t'|} \right] dt'.$$

Thus we have

$$\begin{aligned} |I_{42}(z)| &\lesssim |f|_{L^\infty} \int_{|t'| \leq 1} \frac{dt'}{|\phi| |t'|^3} \\ &\lesssim |f|_{L^\infty} \int_{|t'| \leq 1} \frac{dt'}{(|t_2| + |r(z)|) |t'|^3} \\ &\lesssim |f|_{L^\infty}. \end{aligned}$$

Hence we have the result.

References

- [1] J. Bruna and J. D. Castillo, *Hölder and L^p -estimates for the $\bar{\partial}$ equation in some convex domains with real-analytic boundary*, Math. Ann., **269**(1984), 527-539.
- [2] G. M. Henkin and A. V. Romanov, *Exact Hölder estimates for the solutions of the $\bar{\partial}$ -equation*, Izv. Akad. Nauk SSSR Ser. Mat., **35**(1971), 1171–1183; Math. USSR Izv., **5**(1971), 1180-1192.
- [3] N. Kerzman, *Hölder and L^p estimates for solutions of $\bar{\partial}u = f$ in strongly pseudoconvex domains*, Comm. Pure Appl. Math., **24**(1971), 301-379.
- [4] J. C. Polking, *The Cauchy-Riemann equations in convex domains*, Proc. Symp. Pour Math., **52-3**(1991), 309-322.
- [5] R. M. Range, *Hölder estimates for $\bar{\partial}$ on convex domains in \mathbf{C}^2 with real analytic boundary*, Proc. Symp. Pure Math., **30**(1977), 31-33.
- [6] R. M. Range, *The Carathéodory metric and holomorphic maps on a class of weakly pseudoconvex domains*, Pacific J. Math., **78**(1978), 173-188.

- [7] R. M. Range, *Holomorphic functions and integral representations in several complex variables*, Springer-Verlag, Berlin Heidelberg, New York, 1986.
- [8] R. M. Range, *Integral kernels and Hölder estimates for $\bar{\partial}$ on pseudoconvex domains of finite type in \mathbb{C}^2* , Math. Ann., **288**(1990), 63-74.
- [9] R. M. Range, *On Hölder and BMO estimates for $\bar{\partial}$ on convex domains in \mathbb{C}^2* , J. Geom. Anal., **2**(1992), 575-584.