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Note on the $\bar{\partial}$-problem on the complex ellipsoid

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abstract

Let $D$ be a complex ellipsoid in $\mathbb{C}^n$. In this paper we study Hölder estimates for solutions of the $\bar{\partial}$-problem in $D$.

1. Introduction.

Let $D$ be a complex ellipsoid in $\mathbb{C}^n$. Then $D$ can be written in the following form.

$$D = \{ z : r(z) < 0 \}, \quad r(z) = \sum_{i=1}^{n} |z_i|^{2m_i} - 1,$$

where $m_i (i = 1, \cdots, n)$ are positive integers. We denote by $C_{(0,q)}(\overline{D})$ the space of all $C^1 (0,q)$-forms on $\overline{D}$. We also denote by $\Lambda_{\alpha,(0,q)}(D)$ the space of all $(0,q)$-forms in $D$ whose coefficients are Lipschitz functions of order $\alpha$. Let $M = \max\{2m_i\}$. Let $f$ be a $C^1(0,1)$-form in $\overline{D}$ with $\bar{\partial}f = 0$. Then Range[2] proved that there exists a Lipschitz function $u$ of order $\alpha (\alpha < 1/M)$ in $D$ such that $\bar{\partial}u = f$. On the other hand, Diederich-Fornaess-Wiegerinck[1] obtained Lipschitz solutions of the $\bar{\partial}$-problem in real ellipsoids. In their paper they pointed out that Range's result is still valid in the case where $\alpha = 1/M$. In the present paper we shall prove the following:

**Theorem.** Let $D$ be the complex ellipsoid defined as above. For $f \in C_{(0,q)}^1(\overline{D}), 1 \leq q \leq n$, with $\bar{\partial}f = 0$, there exists $u \in \Lambda_{1/M,(0,q-1)}^1(D)$ such that $\bar{\partial}u = f$.

2. Some lemmas.

Define

$$r_j(z) = \frac{\partial r}{\partial z_j}(z), \quad \Phi(\zeta, z) = \sum_{j=1}^{n} r_j(\zeta)(\zeta_j - z_j).$$

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Further we set
\[ \beta = |\zeta - z|^2, \quad W = \sum_{j=1}^{n} \frac{r_j(\zeta)}{\Phi(\zeta, z)} d\zeta_j, \quad B = \frac{\partial \beta}{\beta} = \sum_{j=1}^{n} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^2} d\zeta_j. \]

For \( \hat{W} = \lambda W + (1 - \lambda)B \), we define
\[ \Omega_q(\hat{W}) = c_{q,n} \hat{W} \wedge (\partial_{\zeta_1 \partial_{\zeta}} \hat{W})^{n-q-1} \wedge (\partial_{\bar{z}} \hat{W})^q, \]
where \( c_{q,n} \) are numerical constants. Now we define for a continuous \((0, q)\)-form \( f(1 \leq q \leq n) \) on \( \overline{D} \)
\[ T^W_q f = \int_{\partial D \times I} f \wedge \Omega_q - \int_{D} f \wedge \Omega_{q-1}(B). \]

Then \( T^W_q f \) satisfies \( \bar{\partial} T^W_q f = f \).

If we set \( \Omega_{q-1}(\hat{W}) = d\lambda \wedge \Omega^{(1)} + \Omega^{(0)} \), then after integrating with respect to \( d\lambda \) we have
\[ \int_{\partial D \times I} f \wedge \Omega_{q-1}(\hat{W}) = \int_{D} f \wedge d\lambda \wedge \Omega^{(1)} = \int_{\partial D} \Omega^{(2)}, \]
where \( \Omega^{(2)} \) is written by using a symbol \( P = \sum_{j=1}^{n} r_j(\zeta) d\zeta_j, Q = \sum_{k=1}^{n} d\bar{\zeta}_k \wedge d\zeta_k, \)
\[ \Omega^{(2)} = \sum_{j=1}^{n-q-1} b_{j,k} \frac{\partial \zeta \wedge P \wedge (\partial_{\zeta} P)^j \wedge Q^{n-q-1-j} \wedge (\sum_{j=1}^{n} d\bar{\zeta}_j \wedge d\zeta_j)^{q-1}}{\Phi^{j+1} \partial_{\zeta}^{n-j-1}}. \]

Range[2] proved the following:

**Lemma 1.** Let \( M = \max_i(2m_i) \). Then it holds that for \((\zeta, z) \in \partial D \times D, \)
\[ |\Phi(\zeta, z)| \geq |\text{Im} \Phi(\zeta, z)| + |r(z)| + \sum_{i=1}^{n} |\zeta_i|^{2m_i-2} |z_i - \zeta_i|^2 + |z - \zeta|^M. \]

Let \( \zeta \in \partial D \). Then \( r_i(\zeta) \neq 0 \) for some \( i \). We may assume without loss of generality that \( i = n \). Then we can choose a small ball \( \tilde{U} \) with center \( \zeta \). We denote by \( U \) a ball with center \( \zeta \) such that \( U \subset \subset \tilde{U} \). By using the partition of unity argument it is sufficient to estimate \( \int_{\partial D \cap U} f \wedge \Omega^{(2)} \). Now we have the following.

**Lemma 2.** For \( z, \zeta \in U \), we define \( x_{2j-1}(\zeta) = \text{Re}(\zeta_j - z_j), x_{2j}(\zeta) = \text{Im}(\zeta_j - z_j), j = 1, \ldots, n-1, y(\zeta) = \text{Im} \Phi(\zeta, z), t(\zeta) = r(\zeta) + |r(z)| \), then \( t, y, x_1, \ldots, x_{2n-2} \) constitute coordinates system in \( U \).
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PROOF. In view of the equality

$$\frac{\partial y}{\partial x_{2j}}(z) = -\frac{1}{2} \frac{\partial r}{\partial x_{2j-1}}(z), \quad \frac{\partial y}{\partial x_{2j-1}}(z) = \frac{1}{2} \frac{\partial r}{\partial x_{2j}}(z),$$

we have

$$\frac{\partial (x_1, \ldots, x_{2n-2}, y, t)}{\partial (x_1, \ldots, x_{2n})} = -2 \left| \frac{\partial r}{\partial \zeta_n} \right|^2 \neq 0.$$ 

This completes the proof of Lemma 2.

We need the following (cf. [1]):

**Lemma 3.** Let $R$ be a positive constant and $j$ a non-negative integer. For $A > 0, q \geq 1$ and $z = x + iy$ it holds that

$$\int_{|z| < R} \frac{|z + w|^j}{(A + |z + w|^j |z|^2)^q} dx dy = \begin{cases} O(A^{1-q}) & (q > 1) \\ O(\log A) & (q = 1). \end{cases}$$

**Proof.** We divide the domain of integration into three parts.

$$\{z : |z| < R\} = \{z : |z| < R, |z| < \frac{1}{2}|w|\}
\cup \{z : |z| < R, |z| \geq \frac{1}{2}|w|, |z + w| < \frac{1}{2}|w|\}
\cup \{z : |z| < R, |z| \geq \frac{1}{2}|w|, |z + w| \geq \frac{1}{2}|w|\}.$$ 

We only estimate

$$I_1 = \int_{|z| < R, |z| < \frac{1}{2}|w|} \frac{|z + w|^j}{(A + |z + w|^j |z|^2)^q} dx dy.$$ 

Using polar coordinates we have

$$I_1 \lesssim \int_{|z| < R} \frac{(\frac{3}{2}|w|)^j}{(A + (\frac{3}{2}|w|)^j |z|^2)^q} dx dy = 2\pi \int_0^R \frac{(\frac{3}{2}|w|)^j}{(A + (\frac{3}{2}|w|)^j r^2)^q} dr.$$ 

Thus we have

$$I_1 = \begin{cases} O(A^{1-q}) & (q > 1) \\ O(\log A) & (q = 1). \end{cases}$$ 

Using similar methods, we can prove the other cases. This completes the proof of Lemma 3.

In order to prove our theorem we use the following Hardy-Littlewood argument.
**Lemma 4.** Let $D$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Then there exists a positive constant $C$ with the following property: If $g$ is a $C^1$ function in $D$ such that for some $K > 0$ and $0 < \alpha < 1$

$$||dg(x)|| \leq K|\text{dist}(x, \partial D)|^{-\alpha} (x \in D),$$

then it holds that

$$|g(x) - g(y)| \leq CK|x - y|^{1-\alpha}(x, y \in D).$$

3. Proof of the theorem.

We set

$$g = \int_{\partial D} f \wedge \Omega^{(2)}. $$

Then we have

$$dg = \int_{\partial D} f \wedge d\Omega^{(2)}.$$

Thus it is sufficient to estimate the following two integrals:

$$I_1 = \int_{\partial D} \left| \frac{\partial \zeta_\beta \wedge P \wedge (\partial \zeta P)^j}{\Phi^{j+1} \beta^{n-j-1}} \right|, \quad I_2 = \int_{\partial D} \left| \frac{P \wedge (\partial \zeta P)^j}{\Phi^{j+1} \beta^{n-j-1}} \right|.$$

We set $x = (t, y, x_1, \cdots, x_{2n-2})$ and $x' = (x_{2j+1}, \cdots, x_{2n-2})$. Then we have by using (2.1)

$$I_1 \leq \int_{|x| < c} \frac{|\zeta_1|^{2m_1-2} \cdots |\zeta_j|^{2m_j-2}dydx_1 \cdots dx_{2n-2}}{(|y| + t + \sum_{i=1}^{n} |\zeta_i|^{2m_i-2} |z_i - \zeta_i|^2 + |z - \zeta|^M)^{j+2}}.$$

We set $\zeta_i - z_i = w_i$. Using Lemma 3 we have

$$I_1 \leq \int_{|x| < c} \frac{|z_1 + w_1|^{2m_1-2} \cdots |z_j + w_j|^{2m_j-2}dx_1 \cdots dx_{2n-2}}{(t + \sum_{i=1}^{n} |z_i + w_i|^{2m_i-2}|w_i|^2 + |x'|^M)^{j+1}}.$$
We set $t^{-1/M}r = u$. Then we have
\[
I_1 \lesssim \int_0^\infty \frac{t^{1/M-1}}{1 + u^M} \, du \lesssim (\text{dist}(z, \partial D))^{1/M-1}.
\]
Next we estimate $I_2$. Following the estimate of $I_1$, we obtain
\[
I_2 \lesssim \int_{|x'|<c} \frac{|\log(t + |x'|^M)| \, dx_{2j+1} \cdots dx_{2n-2}}{(t + |x'|)^{2n-2j-2}} \lesssim \int_0^c \frac{|\log(t + r^M)| \, dr}{r + t}
\]
\[
\lesssim \int_0^c \frac{\log t}{r + t} \, dr \lesssim (\log t)^2.
\]
This completes the proof of the theorem.

References

