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Circular and Necklace Permutations

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Abstract

We enumerate circular and necklace permutations which have beads of several colors, and count the related combinatorial numbers.

0 Introduction

Circular and necklace permutations are standard subjects of high school combinatorics. On the other hand they serve a good introduction to more advanced subject of combinatorics, that is the Pólya theory of counting under group action. In fact these enumerations were historically a precursor of the theory [2, 4, p.558]. In this article we give the enumeration of circular and necklace permuations according to the given number of beads for each colors, and of the related combinatorial numbers.

A *circular permutation* is defined to be an equivalence class of sequences under the cyclic group C_n, i.e., two sequences $a_1a_2\ldots a_n$ and $b_1b_2\ldots b_n$ are equivalent if one is a cyclic shift of the other. A *necklace* (or *necklace permutation*) is defined to be an equivalence class of sequences under the dihedral group D_n, i.e., two sequences $a_1a_2\ldots a_n$ and $b_1b_2\ldots b_n$ are equivalent if one is a cyclic shift of the other or if circular permutaions of them are mutually reflective with respect to some diameter.

We make full use of the counting theory and look back the classical examples, and show how to use Pólya's cycle index efficiently. We use combinatorial notation, especially relative to partitions, as in [4].

1 Cycle indices of cyclic and dihedral groups

We begin with a brief accout of the counting theory under group action. A good introduction to the theory is [3, chapter 6], that is a lecture note without detailed proof . To those who want the proof see [1, chapter 5] or [4, section 7.24].

Let G be a subgroup of the full permutation group S_X of a finite set X. Let $n = |X|$ be the number of elements of X. Each element $g \in G$ is a permutation of X, so g is a product of several disjoint cycles of elements of X: $g = (a_1a_2 \ldots a_k) \ldots (c_1c_2 \ldots c_m)$. The cycle type of g determines a monomial $p_1^{b_1(g)}p_2^{b_2(g)} \ldots p_n^{b_n(g)}$, that is, g has $b_1(g)$ cycles of length 1, $b_2(g)$ cycles of length 2, …in the cycle decomposition, so $(1^{b_1(g)}2^{b_2(g)} \ldots) \vdash n$. *Cycle index* P_G is a polynomial defined by

$$P_G = \frac{1}{|G|} \sum_{g \in G} p_1^{b_1(g)}p_2^{b_2(g)} \ldots p_n^{b_n(g)}.$$

Let $R = \{r, b, \ldots, w\}$ be a set of k colors. Then Pólya's main theorem is that all the inequivalent colorings $f : X \to R$ under G are expressible by a generating function P_G substituted by $p_i = r^i + b^i + \cdots + w^i$.

For circular and necklace permutations let $X = \{1, 2, \ldots n\}$ and let the cyclic group $C_n = \langle a | a^n = e \rangle$ act on X in an obvious way: $a = (1, 2, \ldots, n)$. The coloring $f : X \to R$ is a sequence of length n and then C_n act on the set R^X of sequences as a cyclic shift by $a(r_1r_2 \ldots r_n) = r_2r_3 \ldots r_nr_1$. An equivalence class of the sequences under C_n is exactly a circular permutation. The dihedral group $D_n = \langle a, \varepsilon | a^n = \varepsilon^2 = e, \varepsilon a \varepsilon = a^{-1} \rangle$ acts on X by putting ε being a reflection with respect to some symmetry axis. For instance put

$$\varepsilon = \begin{cases} (n)(1, n-1)(2, n-2) \ldots (\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil) \text{ when } n \text{ odd} \\ (1, n)(2, n-1) \ldots (\frac{n}{2}, \frac{n}{2} + 1) \text{ when } n \text{ even.} \end{cases}$$

An equivalence class of sequences under D_n is a necklace permutation.

Lemma 1 *Cyclic indices of circular and necklace permutations are given by*

$$P_{C_n} = \frac{1}{n} \sum_{d|n} \varphi(d) p_d^{n/d},$$

$$P_{D_n} = \begin{cases} \frac{1}{2n} \left(\sum_{d|n} \varphi(d) p_d^{n/d} + n p_1 p_2^{\lfloor n/2 \rfloor} \right) & \textit{when } n \textit{ odd} \\ \frac{1}{2n} \left(\sum_{d|n} \varphi(d) p_d^{n/d} + \frac{n}{2}(p_2^{n/2} + p_1^2 p_2^{(n/2)-1}) \right) & \textit{when } n \textit{ even,} \end{cases}$$

where $\varphi(n) := \#\{d \in \mathbf{Z} | \gcd(n, d) = 1, 1 \le d \le n\}$ *is the Euler function.*

For proof, notice that $C_n = \bigsqcup_{d|n} C_n(d)$ where $C_n(d)$ is a subset of all the elements of order d and that all elements in $C_n(d)$ have the cycle type $(d^{n/d}) \vdash n$ and $|C_n(d)| = \varphi(d)$. Notice also a coset decomposition: $D_n = C_n \bigsqcup \varepsilon C_n$ and that all the elements of εC_n are reflections. When n is odd all the reflections of εC_n have the cycle type $(1)(2^{\lfloor n/2 \rfloor}) \vdash n$. When n is even we have two kinds of symmetry axes, half the reflections of εC_n have the cycle type $(2^{n/2})$ and another half the reflections have $(1^2)(2^{n/2-1})$.

The total number of circular and necklace permutations of length n with beads of k kinds of colors are given by substituting all $p_i = k$ in cycle indices.

Corollary 2 *The total number $C(n,k)$ of circular permutations of length n with k colors of beads and $N(n,k)$ of necklace permutations are*

$$C(n,k) = \frac{1}{n}\sum_{d|n}\varphi(d)k^{n/d} = \frac{1}{n}\sum_{d|n}\varphi\left(\frac{n}{d}\right)k^d,$$

$$N(n,k) = \frac{1}{2n}\left(\sum_{d|n}\varphi(d)k^{n/d} + \begin{cases} nk^{\lceil n/2 \rceil} & \text{when } n \text{ odd} \\ \frac{n}{2}(k^{n/2} + k^{(n/2)+1}) & \text{when } n \text{ even} \end{cases}\right).$$

We here take the simplest:
Example $(n = 3)$. Cycle indices:

$$P_{C_3} = \frac{1}{3}(p_1^3 + 2p_3), \qquad P_{D_3} = \frac{1}{6}(p_1^3 + 2p_3 + 3p_1p_2).$$

Total numbers: $C(3,k) = (k^3 + 2k)/3, \quad N(3,k) = (k^3 + 2k + 3k^2)/6,$

k	1	2	3	4	5
$C(3,k)$	1	4	11	24	45
$N(3,k)$	1	4	10	20	35

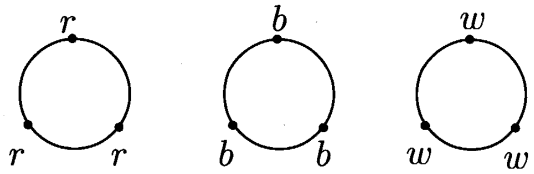
$k = 3$: We have beads of three colors, red r, blue b and white w. Substituting the figure inventory $p_i = r^i + b^i + w^i$ in the cycle index, we get

$$P_{C_3} = \frac{1}{3}\left\{(r+b+w)^3 + 2(r^3 + b^3 + w^3)\right\}$$

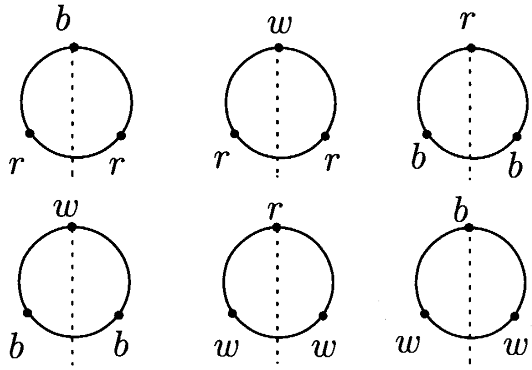
$$= r^3 + b^3 + w^3 + r^2b + r^2w + b^2r + b^2w + w^2r + w^2b + 2rbw$$

by the multinomial expansion.

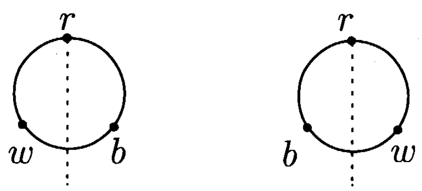
$r^3 + b^3 + w^3$



$r^2b + r^2w + b^2r + b^2w + w^2r + w^2b$



$2rbw$



The first 9 circles are self-reflective, the last 2 circles are pair-reflective each other, with respect to indicated symmetry axes. We can collect terms as

$$\sum r^3 = r^3+b^3+w^3, \ \sum r^2b = r^2b+r^2w+b^2r+b^2w+w^2r+w^2b, \ \sum rbw = rbw.$$

In these sums, for instance $\sum r^2b$ indicates that terms with two beads of one color and one bead of the second color are summed up, r^2b is the typical term of the possible color choices. The sum $\sum rbw$ has just one term when $k = 3$, but has $\binom{k}{3}$ terms when $k \geq 3$ and has no terms hence $= 0$ when $k < 3$. Then we have

$$P_{C_3} = \sum^{k} r^3 + \sum^{k} r^2b + 2\sum^{k} rbw.$$

This expression is independent of the indicated number k of colors. In this example we easily have for necklaces,

$$P_{D_3} = \sum^{k} r^3 + \sum^{k} r^2b + \sum^{k} rbw$$

by computation or by inspection. We will seek their coefficient $[\sum r^\lambda]P_G$, which is the number of essentially inequivalent color patterns under group action.

2 Circular permutations

To get all the circular permutations of length n with beads of k kinds of color, we have to expand P_{C_n} by substituting $p_d = r_1^d + r_2^d + \cdots + r_k^d$, the power sum symmetric polyomials. In this case the expansion is fairly easy. Remember the multinomial expansion and write that in the following way:

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{|\alpha|=n, \alpha\in\mathbf{N}^k} \binom{n}{\alpha} x^\alpha = \sum_{\mu\vdash n, l(\mu)\le k} \binom{n}{\mu} \sum^k x^\mu.$$

Here $l(\mu)$ is the length of a partition μ and $\sum^k x^\mu$ is a monomial symmetric polynomial m_μ of k variables [4, section7.3] defined by

$$\sum^k x^\mu = m_\mu = \sum_\alpha x^\alpha$$

where the last sum runs over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \ldots)$ of the entries of the partition $\mu = (\mu_1, \mu_2, \ldots)$. A monomial symmetric polynomial $\sum^k x^\mu$ collects all the colorings whose choices of the numbers of different k colors of beads are according to the partition $\mu \vdash n$. We have by the substitution

$$P_{C_n} = \frac{1}{n}\sum_{d|n} \varphi(d) p_d^{n/d} = \frac{1}{n}\sum_{d|n} \varphi(d) \left(\sum_{\mu\vdash n/d} \binom{n/d}{\mu} \sum^k r^{d\mu} \right)$$

where $d\mu = (d\mu_1, d\mu_2, \ldots) \vdash d|\mu|$. Putting $\lambda = d\mu \vdash n$, we get the generating function of circular permutations of length n with k colors of beads:

Theorem 3

$$P_{C_n} = \sum_{\lambda\vdash n} \left\{ \frac{1}{n} \sum_{d|\gcd\{\lambda_1,\lambda_2,\ldots\}} \binom{n/d}{\lambda/d} \varphi(d) \right\} \sum^k r^\lambda.$$

where $\lambda/d = (\lambda_1/d, \lambda_2/d, \ldots) \vdash n/d$.

The coefficient $[\sum^k r^\lambda]P_{C_n} = \sum_{d|\gcd\{\lambda_1,\lambda_2,\ldots\}} \binom{n/d}{\lambda/d}\varphi(d)/n$ is the number of essentially different circular permutations according to the number of each color indicated by the partition $\lambda \vdash n$. For instance the number of circular permutations of length $n = 6$ with two red beads, two blue beads and two white beads is $[\sum r^2b^2w^2]P_{C_6} = \{\varphi(1)\binom{6}{2^3} + \varphi(2)\binom{3}{1^3}\}/6 = 16$. A direct counting of this coefficient is possible by the Burnside lemma. The cycle index involves the Burnside lemma beforehand in its polynomial form.

The number of terms in $\sum^k r^\lambda$ is the number of color choices among k colors, according to the partition $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$ which indicates the numbers of each color. The length $l(\lambda)$ is now not greater than k.

Lemma 4 *Let the partition $\lambda \vdash n$ be $\lambda = (1^{b_1}2^{b_2}\ldots)$ and let $l(\lambda) = b_1 + b_2 + \cdots \leq k$. Then the number of terms in a monomial symmetric polynomial $\sum^k r^\lambda$ is*

$$\binom{k}{b_1, b_2, \ldots, k - l(\lambda)} \quad (\textit{possibly} \ = 0).$$

3 Necklaces

As to necklaces we have to expand the correction terms $p_1p_2^{\lfloor n/2\rfloor}$ and $p_1^2p_2^{n/2-1}$ in the cycle index P_{D_n}.

When n is odd we have

$$p_1p_2^{\lfloor n/2\rfloor} = \sum r_1 \left(\sum r_1^2\right)^{(n-1)/2} = \sum r_1 \sum_{\mu \vdash (n-1)/2} \binom{\frac{n-1}{2}}{\mu} \sum r^{2\mu}.$$

For a product of monomial symmetric polynomials, $\sum r_1 \sum r^{2\mu} = \sum_i \sum r^{2\mu+e_i}$ where e_i is i-th unit vector. So

$$p_1p_2^{\lfloor n/2\rfloor} = \sum_{i=1}^{k} \sum_{\mu \vdash (n-1)/2} \binom{\frac{n-1}{2}}{\mu} \sum r^{2\mu+e_i}.$$

Put $\lambda = 2\mu + e_i \vdash n$. Then $\binom{(n-1)/2}{\mu} = \binom{(n-1)/2}{\lfloor\lambda/2\rfloor}$ where $\lfloor\lambda/2\rfloor = (\lfloor\lambda_1/2\rfloor, \lfloor\lambda_2/2\rfloor, \ldots) \vdash (n-1)/2 = \lfloor n/2\rfloor$ and λ runs over all the partitions of n those have exactly one odd entry one by one, in the above summation. Hence

$$p_1p_2^{\lfloor n/2\rfloor} = \sum_\lambda \binom{\lfloor n/2\rfloor}{\lfloor\lambda/2\rfloor} \sum^k r^\lambda$$

where the sum runs over all the partitions λ of n those have exactly one odd entry. When n is odd, the generating function of necklaces is

$$P_{D_n} = \frac{1}{2n}\left(\sum_{d|n}\varphi(d)p_d^{n/d} + np_1p_2^{\lfloor n/2\rfloor}\right)$$

$$= \frac{1}{2}\left\{\sum_{\lambda\vdash n}\left(\frac{1}{n}\sum_{d|\gcd\{\lambda_1,\lambda_2,\ldots\}}\binom{n/d}{\lambda/d}\varphi(d)\right)\sum^{k} r^{\lambda} + \sum_{\lambda}\binom{\lfloor n/2\rfloor}{\lfloor\lambda/2\rfloor}\sum^{k} r^{\lambda}\right\}$$

$$= \sum_{\lambda}^{(1)}\frac{1}{2}\left\{\frac{1}{n}\sum_{d|\gcd\lambda}\binom{n/d}{\lambda/d}\varphi(d) + \binom{\lfloor n/2\rfloor}{\lfloor\lambda/2\rfloor}\right\}\sum^{k} r^{\lambda} + \sum_{\lambda}^{(2)}\frac{1}{2n}\left(\sum_{d|\gcd\lambda}\binom{n/d}{\lambda/d}\varphi(d)\right)\sum^{k} r^{\lambda}$$

where the first sum $\sum^{(1)}$ runs over partitions $\lambda \vdash n$, $l(\lambda) \leq k$ which have the exactly one odd entries, the second sum $\sum^{(2)}$ runs over the other partitions $\lambda \vdash n$, $l(\lambda) \leq k$ and gcd $\lambda = \gcd\{\lambda_1, \lambda_2, \ldots\}$ is the great common divisor of nonzero entries of λ.

When n is even direct expansion of $p_1^2 p_2^{n/2-1}$ is possible but clumsy. We quote the following proposition which is convenient for power sum expansions in the cycle index. Define the power sum symmetric polynomials as

$$p_n = m_n = \sum_i x_i^n, \quad n \geq 1 \quad \text{with } p_0 = 1$$

$$p_\mu = p_{\mu_1}p_{\mu_2}\cdots \qquad \text{if } \mu = (\mu_1, \mu_2, \ldots).$$

Proposition 5 ([4], 7.7.1) *Let* $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$, *where* $l = l(\lambda)$, *and set*

$$p_\lambda = \sum_{\mu\vdash n} R_{\lambda\mu} m_\mu.$$

Let $k = l(\mu)$. *Then* $R_{\lambda\mu}$ *is the number of ordered partitions* $\pi = (B_1, \ldots, B_k)$ *of the set* $[l]$ *such that*

$$\mu_j = \sum_{i\in B_j}\lambda_i, \; 1 \leq j \leq k.$$

We then have

$$p_1^2 p_2^{n/2-1} = p_{(1^2 2^{n/2-1})} = \sum_{\lambda} R_{(1^2 2^{n/2-1}),\lambda}\, m_\lambda.$$

If $R_{\mu\lambda} \neq 0$ then μ is a refinement of λ. So if $R_{(1^2 2^{n/2-1}),\lambda} \neq 0$ then either all entries of λ are even or exactly two entries of λ are odd. We get $R_{(1^2 2^{n/2-1}),\lambda} = \binom{n/2}{\lambda/2}$ when all entries of λ are even and $R_{(1^2 2^{n/2-1}),\lambda} = 2\binom{n/2-1}{\lfloor\lambda/2\rfloor}$ when λ has exactly two odd entries, as an exercise of prop 5 (though the whole article is an exercise of the counting theory). We have

$$p_2^{n/2} + p_1^2 p_2^{(n/2)-1} = 2\sum_{\lambda}^{(3)} \binom{n/2}{\lambda/2} \sum r^\lambda + 2\sum_{\lambda}^{(4)} \binom{n/2-1}{\lfloor\lambda/2\rfloor} \sum r^\lambda$$

where $\sum^{(3)}$ runs over partitions $\lambda \vdash n, l(\lambda) \leq k$ which have all even entries and $\sum^{(4)}$ runs over partitions $\lambda \vdash n, l(\lambda) \leq k$ which have exactly two odd entries. Substituting the power sums and gathering terms together in

$$P_{D_n} = \frac{1}{2n}\left(\sum_{d|n} \varphi(d) p_d^{n/d} + \frac{n}{2}(p_2^{n/2} + p_1^2 p_2^{(n/2)-1})\right),$$

we get the generating function of necklaces of length n with beads of k kinds of color.

Theorem 6 *When n is odd,*

$$P_{D_n} = \sum_{\lambda}^{(1)} \frac{1}{2}\left\{\frac{1}{n}\sum_{d|\gcd\lambda} \varphi(d)\binom{n/d}{\lambda/d} + \binom{\lfloor n/2\rfloor}{\lfloor\lambda/2\rfloor}\right\}\sum^{k} r^\lambda + \sum_{\lambda}^{(2)}\left(\frac{1}{2n}\sum_{d|\gcd\lambda}\varphi(d)\binom{n/d}{\lambda/d}\right)\sum^{k} r^\lambda.$$

When n is even,

$$P_{D_n} = \sum_{\lambda}^{(3)} \frac{1}{2}\left\{\frac{1}{n}\sum_{d|\gcd\lambda} \varphi(d)\binom{n/d}{\lambda/d} + \binom{n/2}{\lambda/2}\right\}\sum^{k} r^\lambda +$$

$$\sum_{\lambda}^{(4)} \frac{1}{2}\left\{\frac{1}{n}\sum_{d|\gcd\lambda} \varphi(d)\binom{n/d}{\lambda/d} + \binom{\frac{n}{2}-1}{\lfloor\lambda/2\rfloor}\right\}\sum^{k} r^\lambda + \sum_{\lambda}^{(5)}\left(\frac{1}{2n}\sum_{d|\gcd\lambda}\varphi(d)\binom{n/d}{\lambda/d}\right)\sum^{k} r^\lambda.$$

Here all the summations are for partitions $\lambda \vdash n, l(\lambda) \leq k$ with the additional conditions: $\sum^{(1)}$ runs over λ those have exactly one odd entry, $\sum^{(3)}$ runs over λ those have all even entries, $\sum^{(4)}$ runs over λ those have exactly two odd entries, $\sum^{(2)}$ and $\sum^{(5)}$ run over the remaining partitions.

The coefficient $[\sum r^\lambda]P_{D_n}$ is the number of essentially different necklaces according to $\lambda \vdash n$ which shows the number of beads of each colors. For instance the number of necklaces with 2 red beads, 2 blue beads and 2 white beads is in case $\sum^{(3)}$ and so $[\sum r^2b^2w^2]P_{D_3} = \{16 + \binom{3}{1^3}\}/2 = 11$ (16 is the number of circular permutations already counted). We count this coefficient by the Burnside lemma in the last section. We find that the correction term of each generating function corresponds to the number of self-reflective circular permutations relative to the symmetry axes of each case. Try examples when $n = 6$ or 12, and enjoy the correspondence with necklaces and mathematical expressions, in the Pólya theory!

4 Burnside Lemma and Examples

We will give a direct counting by the Burnside lemma of the number of circular permutations and necklaces varying the number of beads for each colors. The Burnside(-Cauchy-Frobenius) lemma is the following, which enables us to replace counting equivalence classes by counting fixed points of the permutation group.

Lemma 7 ([4], 7.24.5 or [1], Theorem 5.2) *Let X be a finite set and G a subgroup of the full permutation group S_X. For each $g \in G$ let*

$$\mathrm{Fix}(g) = \{x \in X | gx = x\}$$

be the fixed point set of the permutation g. Let X/G be the set of orbits of G. Then

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}(g)|.$$

In other words, the average number of elements of X fixed by an elements of G is equal to the number of equivalence classes.

Let $R = \{r_1, r_2, \ldots\}$ be a set of colors and let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$ be a partition which shows the number of beads for each color. Let $R^\lambda = \{s = s_1s_2\ldots s_n \in R^{[n]} | s$ contains λ_1 r_1's, λ_2 r_2's,$\ldots\}$ be the set of all the sequences of length n in which color r_1 appears λ_1-times, color r_2 appears λ_2-times and so on. Then $G = C_n$(or D_n respectively) acts on $X = R^\lambda$, and $X/G = R^\lambda/G$ is the set of all the circular permutations of length n which have λ_1-beads of color r_1, λ_2-beads of color r_2 and so on when $G = C_n$ (or the set of all the necklaces which have the same numbers of beads for each color when $G = D_n$ respectively). The number of elements of these sets are obtained as coefficients $[r^\lambda]P_G = |R^\lambda/G|$ of generating functions in section 2 and 3. We apply here the Burnside lemma 7.

Corollary 8 *The number of circular permutations of length n which have λ_1-beads of color r_1, λ_2-beads of color r_2 and so on, is given by*

$$[r^\lambda]P_{C_n} = \frac{1}{n} \sum_{d|\gcd\{\lambda_1,\lambda_2,\ldots\}} \binom{n/d}{\lambda/d} \varphi(d).$$

Proof. We know

$$[r^\lambda]P_{C_n} = |R^\lambda/C_n| = \frac{1}{n} \sum_{g\in C_n} |\mathrm{Fix}(g)| = \frac{1}{n} \sum_{d|n} \sum_{g\in C_n(d)} |\mathrm{Fix}(g)|$$

by the Burnside lemma and

$$C_n = \bigsqcup_{d|n} C_n(d)$$

where $C_n(d)$ is a subset of all the elements of order d and all elements in $C_n(d)$ have the cycle type $(d^{n/d}) \vdash n$. Let $g \in C_n(d)$ and $s = s_1 s_2 \ldots s_n \in \mathrm{Fix}(g, R^\lambda)$. Then for cycles $(i_1, i_2, \ldots, i_d)$ of the permutation g, the corresponding elements in the sequence s must have the same color: $s_{i_1} = s_{i_2} = \cdots = s_{i_d}$ since g fixes s. Hence we see $d|\lambda_i$ $(\forall i)$, and get $|\mathrm{Fix}(g, R^\lambda)| = \binom{n/d}{\lambda/d}$ by considering selections of colors to the (n/d)-cycles in g. Since $|C_n(d)| = \varphi(d)$ result follows.

Corollary 9 *The number $[r^\lambda]P_{D_n}$ of necklaces which have λ_1-beads of color r_1, λ_2-beads of color r_2 and so on, is given as follows.*
When n is odd, we have

$$[r^\lambda]P_{D_n} = \frac{1}{2n} \left\{ \sum_{d|\gcd\,\lambda} \varphi(d) \binom{n/d}{\lambda/d} + n \binom{(n-1)/2}{\lfloor \lambda/2 \rfloor} \right\}$$

if λ has exactly one odd entry, and

$$[r^\lambda]P_{D_n} = \frac{1}{2n} \sum_{d|\gcd\,\lambda} \varphi(d) \binom{n/d}{\lambda/d}$$

for other λ.
When n *is even, we have*

$$[r^{\lambda}]P_{D_n} = \frac{1}{2n}\left\{\sum_{d|\gcd\,\lambda}\varphi(d)\binom{n/d}{\lambda/d} + n\binom{n/2}{\lambda/2}\right\}$$

if λ *has all even entries,*

$$[r^{\lambda}]P_{D_n} = \frac{1}{2n}\left\{\sum_{d|\gcd\,\lambda}\varphi(d)\binom{n/d}{\lambda/d} + n\binom{(n-2)/2}{\lfloor\lambda/2\rfloor}\right\}$$

if λ *has exactly two odd entries, and*

$$[r^{\lambda}]P_{D_n} = \frac{1}{2n}\sum_{d|\gcd\,\lambda}\varphi(d)\binom{n/d}{\lambda/d}$$

for other λ.

Proof. We have to take reflections into consideration. We know

$$[r^{\lambda}]P_{D_n} = |R^{\lambda}/D_n| = \frac{1}{2n}\sum_{g\in D_n}|\mathrm{Fix}(g)| = \frac{1}{2n}\left(\sum_{g\in C_n}|\mathrm{Fix}(g)| + \sum_{g\in\varepsilon C_n}|\mathrm{Fix}(g)|\right)$$

by the Burnside lemma and a coset decomposition

$$D_n = C_n \bigsqcup \varepsilon C_n$$

where $D_n = \langle a, \varepsilon | a^n = \varepsilon^2 = e, \varepsilon a \varepsilon = a^{-1}\rangle$, $C_n = \langle a \rangle$ is a subgroup of all the rotations of D_n, and all the elements of εC_n are reflections with respect to some axes. When n is odd all the reflections of εC_n have the cycle type $(1)(2^{(n-1)/2}) \vdash n$. When n is even we have two kinds of symmetry axes, half the reflections of εC_n have the cycle type $(2^{n/2})$ and another half the reflections have $(1^2)(2^{(n-2)/2})$.
Take a case when n is even for instance. Then half the reflections $\varepsilon C_n^2 = \{\varepsilon, \varepsilon a^2, \ldots, \varepsilon a^{n-2}\}$ of εC_n have the cycle type $(2^{n/2})$. The typical element is

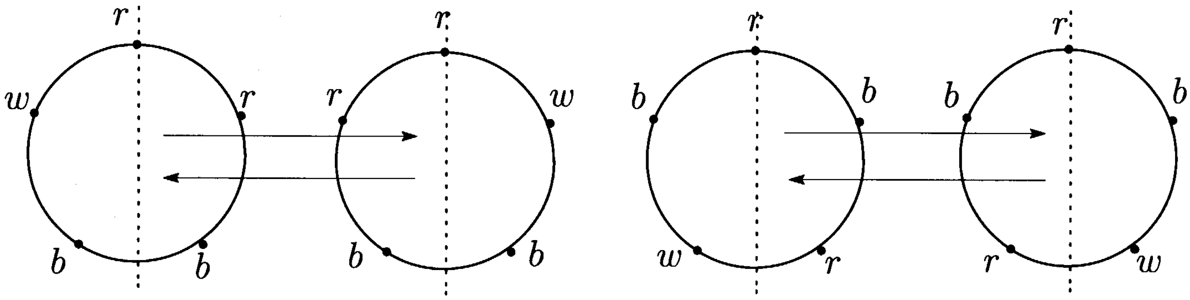
$$\varepsilon = (1, n)(2, n-1)\ldots\left(\frac{n}{2}, \frac{n}{2}+1\right).$$

Let $s = s_1 s_2 \ldots s_n \in \mathrm{Fix}(\varepsilon, R^{\lambda})$. Then $s_1 = s_n, s_2 = s_{n-1}, \ldots, s_{n/2} = s_{(n/2)+1}$. Hence $2|\lambda_i$ $(\forall i)$. If $g \in \varepsilon C_n^2$, then we have $|\mathrm{Fix}(g, R^{\lambda})| = \binom{n/2}{\lambda/2}$ for λ which has all even entries and $\mathrm{Fix}(g, R^\lambda) = \emptyset$ for λ which has an odd entry. Another half $\varepsilon a C_n^2 = \{\varepsilon a, \varepsilon a^3, \ldots, \varepsilon a^{n-1}\}$ of εC_n have the cycle type $(1^2)(2^{(n-2)/2})$. The typical element is

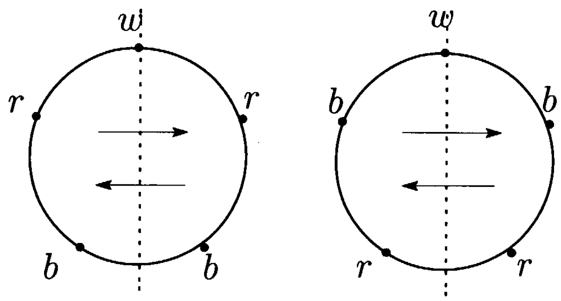
$$\varepsilon a = (n)(1, n-1)(2, n-2)\ldots\left(\frac{n}{2}-1, \frac{n}{2}+1\right)\left(\frac{n}{2}\right).$$

The same reasoning shows that if $g \in \varepsilon C_n^2$, then $|\mathrm{Fix}(g, R^\lambda)| = \binom{n/2}{\lambda/2}$ for λ which has all even entries, $|\mathrm{Fix}(g, R^\lambda)| = 2\binom{(n-2)/2}{\lfloor \lambda/2 \rfloor}$ for λ which has exactly two odd entries, and $\mathrm{Fix}(g, R^\lambda) = \emptyset$ for other λ. Results for even n now follow. The simillar argument goes through the case when n is odd.

Examples. When $n = 5$ and $\lambda = (2, 2, 1)$, the corresponding circular permutations are, $[r^2b^2w]P_{C_5} = \frac{1}{5}\binom{5}{2,2,1} = \frac{30}{5} = 6$ as



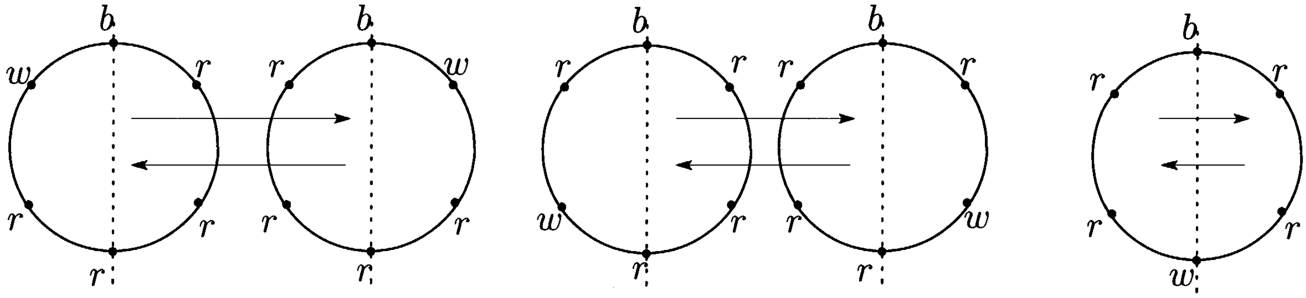
pair-wise reflective



self-reflective

On the other hand the corresponding necklaces are of the number $[r^2b^2w]P_{D_5} = \frac{1}{2}\left\{\frac{1}{5}\binom{5}{2,2,1} + \binom{2}{1}\right\} = \frac{1}{2}(6+2) = 4$. We see that the correction term $\binom{2}{1} = 2$ is the number of self-reflective circular permutations.

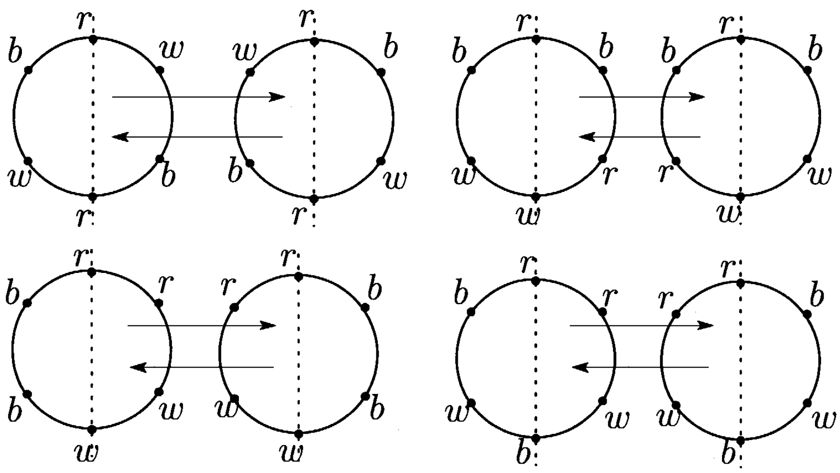
When $n = 6$, first let $\lambda = (4, 1, 1)$. Then the corresponding circular permutations are, $[r^4bw]P_{C_6} = \frac{1}{6}\binom{6}{4,1,1} = \frac{30}{6} = 5$ as



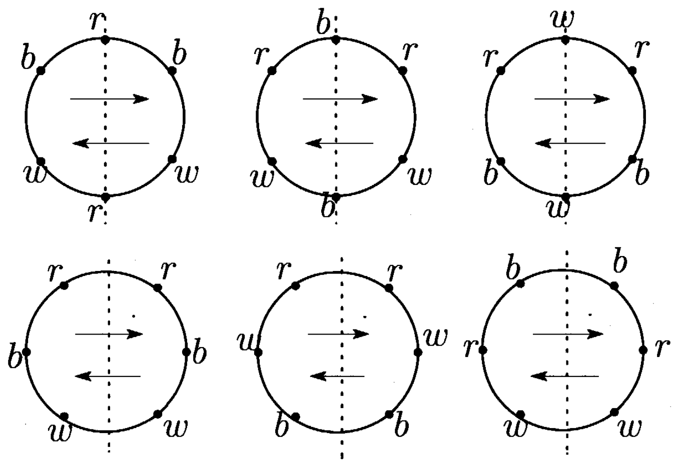
pair-wise reflective

self-reflective

The corresponding necklaces are of the number $[r^4bw]P_{D_6} = \frac{1}{2}\left\{\frac{1}{6}\binom{6}{4,1,1} + \binom{2}{2}\right\} = \frac{1}{2}(5+1) = 3$. Next let $\lambda = (2,2,2) \vdash 6$. Then $[r^2b^2w^2]P_{C_6} = \frac{1}{6}\left\{\binom{6}{2,2,2} + \binom{3}{1,1,1}\right\} = \frac{1}{6}(90+6) = 16$ as



pair-wise reflective



self-reflective

And $[r^2b^2w^2]P_{D_6} = \frac{1}{2}\left[\frac{1}{6}\left\{\binom{6}{2,2,2} + \binom{3}{1,1,1}\right\} + \binom{3}{1,1,1}\right] = \frac{1}{2}(16+6) = 11$. We see again that the correction terms $\binom{2}{2} = 1$ and $\binom{3}{1,1,1} = 6$ are the numbers of self-reflective permutations respectively. Incidentally here we can ask, is there efficient way of writing out all the color patterns one by one in $[r^\lambda]P_G$?

The observation of examples and the proof of corollary 9 show that the correction term counts the number of self-reflective, that is fixed by some reflexion, circular permutations.

Theorem 10 *The number of self-reflective permutations among circular permutations of length n which have λ_1-beads of color r_1, λ_2-beads of color r_2 and so on, is given by*

$$2[r^\lambda]P_{D_n} - [r^\lambda]P_{C_n}$$

$$= \begin{cases} \binom{\lfloor n/2 \rfloor}{\lfloor \lambda/2 \rfloor} & \textit{when } n \textit{ is odd and } \lambda \textit{ has only one odd entry} \\ \binom{n/2}{\lambda/2} & \textit{when } n \textit{ is even and } \lambda \textit{ has all even entries} \\ \binom{(n-2)/2}{\lfloor \lambda/2 \rfloor} & \textit{when } n \textit{ is even and } \lambda \textit{ has exactly two odd entries} \\ 0 & \textit{otherwise.} \end{cases}$$

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