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$H^p$ Extensions of Holomorphic Functions from Submanifolds of a Strictly Pseudoconvex Domain with Non-Smooth Boundary

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Abstract

We prove $H^p$ ($1 < p < \infty$) extensions of holomorphic functions from submanifolds of a strictly pseudoconvex domain in $\mathbb{C}^n$ with non-smooth boundary.

1 Introduction

Let $D \subset \subset \mathbb{C}^n$ be a strictly pseudoconvex domain (with not necessarily smooth boundary) and let $X$ be a closed complex submanifold of some neighborhood of $\overline{D}$. Then Henkin-Leiterer [HER] proved that for any bounded holomorphic function $f$ in $X \cap D$, there exists a bounded holomorphic function $g$ in $D$ such that $f = g$ on $X \cap D$. Moreover, if $f$ is holomorphic in $X \cap D$ that is continuous on $X \cap \overline{D}$, then there exists a holomorphic function $g$ in $D$ that is continuous on $\overline{D}$ such that $f = g$ on $X \cap D$. On the other hand the author [AD2] proved that for any $L^p$ ($1 \leq p < \infty$) holomorphic function $f$ in $X \cap D$, there exists an $L^p$ holomorphic function $g$ in $D$ such that $f = g$ on $X \cap D$. In this paper, we show that any $L^p$ ($1 < p < \infty$) holomorphic function in $X \cap D$ can be extended to an $H^p$ function in $D$ under the assumption that the defining function for $D$ is of class $C^3$.

Theorem 1 Let $D$ be a strictly pseudoconvex domain in $\mathbb{C}^n$ with non-smooth boundary. Assume that the defining function for $D$ is of class $C^3$. Let $X$ be a closed complex submanifold in a neighborhood $D$ of $D$. Let $1 < p < \infty$ and let $f$ be an $L^p$ holomorphic function in $X \cap D$. Then there exists an $H^p$ function $F$ in $D$ such that $F(z) = f(z)$ for $z \in X \cap D$.

Remark 1 Suppose that $D \subset \subset \mathbb{C}^n$ is a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and that $X$ intersects $\partial D$ transversally. Then Theorem 1 was first proved by Cumenge [CUM] and then by Beatrous [BEA] for $1 \leq p < \infty$. The bounded and continuous extensions of holomorphic functions from $X \cap D$ to $D$ were first proved by Henkin [HEN].

2 Preliminaries

Let $D \subset \subset \mathbb{C}^n$ be a strictly pseudoconvex open set and let $\rho$ be a strictly plurisubharmonic $C^3$ function in a neighborhood $\theta$ of $\partial D$ such that

$D \cap \theta = \{z \in \theta \mid \rho(z) < 0\}$. 
Define \( N(\rho) = \{ z \in \theta \mid \rho(z) = 0 \} \). Assume that \( N(\rho) \subset \subset \theta \). Define

\[
F(z, \zeta) = 2 \sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j} (\zeta_j - z_j) - \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_j \partial \zeta_k} (\zeta_j - z_j)(\zeta_k - z_k).
\]

Then Henkin-Leiterer [HER] proved the following:

**Proposition 1** There exist a positive number \( \varepsilon \), a neighborhood \( U \subset \subset \theta \) of \( N(\rho) \) and \( C^1 \) functions \( \Phi(z, \zeta), \tilde{\Phi}(z, \zeta), M(z, \zeta) \) and \( \tilde{M}(z, \zeta) \) for \( \zeta \in U \) and \( z \in U \cup D \) such that the following conditions are fulfilled:

(i) There exists a constant \( \beta > 0 \) such that

\[
\text{Re} F(z, \zeta) \geq \rho(\zeta) - \rho(z) + \beta |\zeta - z|^2
\]

for \( \zeta, z \in \bar{\theta}, |\zeta - z| \leq 2\varepsilon \).

(ii) \( \Phi(z, \zeta) \) and \( \tilde{\Phi}(z, \zeta) \) depend holomorphically on \( z \in U \cup D \).

(iii) \( \Phi(z, \zeta) \neq 0 \) and \( \tilde{\Phi}(z, \zeta) \neq 0 \) for \( \zeta \in U, z \in D \cup U \) with \( |\zeta - z| \geq \varepsilon \). \( M(z, \zeta) \neq 0 \) and \( \tilde{M}(z, \zeta) \neq 0 \) for \( \zeta \in U, z \in D \cup U \);

\[
\Phi(z, \zeta) = F(z, \zeta)M(z, \zeta) \quad \text{and} \quad \tilde{\Phi}(z, \zeta) = (F(z, \zeta) - 2\rho(\zeta))\tilde{M}(z, \zeta)
\]

for \( \zeta \in U, z \in D \cup U \) with \( |\zeta - z| \leq \varepsilon \).

(iv) \( \tilde{\Phi}(z, \zeta) = \Phi(z, \zeta) \) for \( \zeta \in N(\rho), z \in U \cup D \).

(v) Let \( V_1 \) be a neighborhood of \( N(\rho) \) such that \( V_1 \cup D \) is strictly pseudoconvex and \( V_1 \subset \subset U \). Then there exist the \( C^1 \) map \( w = (w_1, \cdots, w_n) : (V_1 \cup D) \times V_1 \rightarrow \mathbb{C}^n, \) holomorphic in \( z \in V_1 \cup D, \) and

\[
<w(z, \zeta), \zeta - z> = \Phi(z, \zeta),
\]

where we define

\[
<w, w> = \sum_{j=1}^{n} w_j \bar{w}_j
\]

for \( w = (w_1, \cdots, w_n) \), \( w = (w_1, \cdots, w_n) \in \mathbb{C}^n \).

We choose a neighborhood \( V_2 \) of \( N(\rho) \) such that \( V_2 \subset \subset V_1 \) and a \( C^\infty \) function \( \chi \) on \( \mathbb{C}^n \) such that

\[
\chi(z) = \begin{cases} 
0 & (z \in \mathbb{C}^n \setminus V_1) \\
1 & (z \in V_2)
\end{cases}
\]

**Definition 1** For any \( L^p \) (\( p \geq 1 \)) function \( f \), define

\[
L_D f(z) = \frac{n!}{(2\pi i)^n} \int_{D} f(\zeta) \left( \frac{\chi(\zeta)w_j(z, \zeta)}{\Phi(z, \zeta)} \right)^j \wedge \omega(\zeta),
\]

where \( \omega(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_n \).

Henkin-Leiterer [HER] proved the following:
Proposition 2 If $f$ is $L^p$ ($1 \leq p \leq \infty$) holomorphic in $D$, then we have
\[ f(z) = L_D f(z) \]
for $z \in D$.

We set $X = \{ z \in \mathbb{C}^n \mid z_n = 0 \}$. For $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{C}^n$ we write $\zeta' = (\zeta_1, \cdots, \zeta_{n-1})$.

Define
\[ \bar{\partial}_\zeta = \sum_{j=1}^{n-1} \frac{\partial}{\partial \zeta_j} \bar{d}_{\zeta_j}, \quad \partial_\zeta = \sum_{j=1}^{n-1} \frac{\partial}{\partial \zeta_j} d_{\zeta_j}, \]
\[ d_{\zeta'} = \bar{\partial}_\zeta + \partial_\zeta', \quad \omega_{\zeta'}(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_{n-1}. \]

Moreover, we define
\[ w'(z, \zeta) = (w_1(z, \zeta), \cdots, w_{n-1}(z, \zeta)), \]
\[ \tilde{\omega}_{\zeta'} \left( \frac{\chi(\zeta)w'(z, \zeta)}{\Phi(z, \zeta)} \right) = \prod_{j=1}^{n-1} \partial_\zeta' \left( \frac{\chi(\zeta)w_j(z, \zeta)}{\Phi(z, \zeta)} \right). \]

By the construction of $\Phi(z, \zeta)$, there exists a neighborhood $U_{\partial D \setminus X}$ of $\partial D \setminus X$ such that $\Phi(z, \zeta) \neq 0$ for $\zeta \in X \cap \overline{D}$, $z \in D \cup U_{\partial D \setminus X}$. For every $L^p$ holomorphic function $f$ in $X \cap D$ and $z \in D \cup U_{\partial D \setminus X}$, define
\[ E_f(z) = \frac{(n-1)!}{(2\pi i)^{n-1}} \int_{X \cap D} f(\zeta) \tilde{\omega}_{\zeta'} \left( \frac{\chi(\zeta)w'(z, \zeta)}{\Phi(z, \zeta)} \right) \wedge \omega_{\zeta'}(\zeta). \]

The following proposition follows from Proposition 2.

Proposition 3 $E_f$ is holomorphic in $D \cup U_{\partial D \setminus X}$ and $f(z) = E_f(z)$ for $z \in D \cap X$.

For $z \in V_2 \cup D$, $\zeta \in V_2 \cap D$, define
\[ \Phi^*(z, \zeta) = \Phi(z, \zeta), \quad w^*(z, \zeta) = -w(z, \zeta), \]
\[ (w^*(z, \zeta))' = (w^*_1(z, \zeta), \cdots, w^*_{n-1}(z, \zeta)). \]

Then $\Phi^*(z, \zeta) \neq 0$ and $\Phi(z, \zeta) \neq 0$ for $z \in \partial D \setminus X$, $\zeta \in X \cap \overline{D}$. Consequently, for every fixed $z \in \partial D \setminus X$,
\[ \det_{1,n-1} \left( \frac{w^*(z, \zeta)}{\Phi^*(z, \zeta)}, \bar{\partial}_\zeta' \frac{\chi(\zeta)w(z, \zeta)}{\Phi(z, \zeta)} \right) \]
is continuous on $\overline{D} \cap X$. By Henkin-Leiterer [HER] we have the following:

Proposition 4 For every $L^p$ ($1 \leq p \leq \infty$) holomorphic function $f$ in $X \cap D$ and all $z \in \partial D \setminus X$, we have
\[ Ef(z) = z^n \frac{(-1)^n}{(2\pi i)^{n-1}} \int_{\zeta \in X \cap D} f(\zeta) \det_{1,n-1} \left( \frac{w^*(z, \zeta)}{\Phi^*(z, \zeta)}, \bar{\partial}_\zeta' \frac{\chi(\zeta)w(z, \zeta)}{\Phi(z, \zeta)} \right) \wedge \omega_{\zeta'}(\zeta). \]
Define
\[ dV_{n-1}(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_{n-1}. \]
We write
\[ K(z, \zeta) dV_{n-1}(\zeta) = z_n \frac{(-1)^n}{(2\pi i)^n} \det_{1,n-1} \left( \frac{w^*(z, \zeta)}{\Phi^*(z, \zeta)}, \frac{\chi(z) w(z, \zeta)}{\Phi(z, \zeta)} \right) \wedge \omega^*_\zeta(\zeta). \]

It follows from Proposition 4 that for any $L^p$ ($1 \leq p \leq \infty$) holomorphic function $f$ in $X \cap D$ and any $z \in \partial D \setminus X$, we have
\[ Ef(z) = \int_{X \cap D} f(\zeta) K(z, \zeta) dV_{n-1}(\zeta). \]

**Definition 2** We denote by $S^{\text{reg}}$ the smooth part of $\partial D$.

We first define the Hardy space $H^p(D)$ ($0 < p \leq \infty$) for a bounded domain in $\mathbb{C}^n$ with smooth boundary.

**Definition 3** Let $D$ be a bounded domain in $\mathbb{C}^n$ with smooth boundary and let $\rho$ be a defining function for $D$. For $\delta > 0$, define $D_\delta = \{ z \mid \rho(z) < -\delta \}$. We say that $f$ belongs to $H^p(D)$ ($0 < p < \infty$) if $f$ is holomorphic in $D$ and
\[ \sup_{\delta > 0} \int_{\partial D_\delta} |f(\zeta)|^p d\sigma_\delta < \infty, \]
where $d\sigma_\delta$ is the surface measure on $\partial D_\delta$. We say that a holomorphic function $f$ belongs to $H^\infty(D)$ if $\sup_{z \in D} |f(z)| < \infty$.

Suppose $D$ is a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. We set for sufficiently small $\delta_0 > 0$,
\[ F_{\delta_0} = \{ z + \alpha \nu_z \mid z \in \partial D \cap X, \ \delta_0 > \alpha > 0 \}, \]
where $\nu_z$ is the unit inward normal vector at $z$ for $\partial D$. If
\[ \int_{\partial D \setminus X} |Ef(z)|^p < \infty, \]
then there exists a constant $C > 0$ such that for sufficiently small $\delta$ and $\delta_1$ ($0 < \delta < \delta_1$),
\[ \int_{\partial D_{\delta_1}} |Ef(z)|^p d\sigma_{\delta_1} \leq C \int_{\partial D_\delta} |Ef(z)|^p d\sigma_\delta 
\]
\[ = C \int_{\partial D_\delta \setminus F_{\delta_0}} |Ef(z)|^p d\sigma_\delta \rightarrow C \int_{\partial D \setminus X} |Ef(z)|^p d\sigma \]
as $\delta \to 0$, which implies that $Ef \in H^p(D)$.

Next suppose that $D$ is a strictly pseudoconvex domain in $\mathbb{C}^n$ with non-smooth boundary. Then the set $\partial D \setminus S^{\text{reg}}$ is locally contained in a real $C^1$ submanifold of real dimension
≤ n (see Theorem 1.4.21, Henkin-Leiterer [HER]). Thus \( X \cap S^{reg} \) has measure 0 for the surface measure \( d\sigma \). Hence we have

\[
\int_{S^{reg}} |Ef(z)|^p d\sigma = \int_{S^{reg} \setminus X} |Ef(z)|^p d\sigma.
\]

Therefore, in case \( D \) is a strictly pseudoconvex domain with non-smooth boundary, we define as follows:

**Definition 4** We say that \( Ef \) belongs to \( H^p(D) \) \((0 < p < \infty)\) if

\[
\int_{S^{reg} \setminus X} |Ef(z)|^p d\sigma < \infty.
\]

By Henkin-Leiterer [HER], there exists a constant \( C > 0 \) such that

\[
\left\| \det_{1,n-1} \left( \frac{w^*}{\Phi^*}, \frac{\Phi}{\Phi^*}, \frac{x_w}{\Phi} \right) \right\| \leq C \left\{ \frac{1}{|\zeta - z|^{2n-1}} + \frac{\|dp(z)\|}{|\Phi||\Phi^*|} \frac{1}{|\zeta - z|^{2n-4}} + \frac{\|d\zeta^i \rho(z)\|^2}{|\Phi^2|\Phi^*|} \frac{1}{|\zeta - z|^{2n-5}} + \frac{\|d\zeta^i \rho(z)\| \|\frac{\partial p(z)}{\partial z_n(z)}\|}{|\Phi^2|\Phi^*|} \frac{1}{|\zeta - z|^{2n-5}} \right\}.
\]

We set

\[
K_1(z, \zeta) = \frac{|z_n|}{|\zeta - z|^{2n-1}},
\]

\[
K_2(z, \zeta) = \frac{|z_n| \|dp(z)\|}{|\Phi(z, \zeta)||\Phi^*(z, \zeta)||\zeta - z|^{2n-4}},
\]

\[
K_3(z, \zeta) = \frac{|z_n| \|d\zeta^i \rho(z)\|^2}{|\Phi(z, \zeta)||\Phi^*(z, \zeta)||\zeta - z|^{2n-5}},
\]

\[
K_4(z, \zeta) = \frac{|z_n| \|d\zeta^i \rho(z)\| \|\frac{\partial p(z)}{\partial z_n(z)}\|}{|\Phi(z, \zeta)||\Phi^*(z, \zeta)||\zeta - z|^{2n-5}}.
\]

For \( \delta > 0 \) sufficiently small, define

\[
Ef(z) := \int_{X \cap D} |f(\zeta)| K_i(z, \zeta) dV_{n-1}(\zeta) \quad (i = 1, 2, 3, 4).
\]

Henkin-Leiterer (Lemma 3.6.6 [HER]) proved the following:

**Lemma 1** There is a constant \( C > 0 \) such that for all \( z \in \partial D \setminus X \), the following estimates hold:

\[
\int_{\zeta \in X \cap D \setminus V_2} K_i(z, \zeta) dV_{n-1} \leq C
\]

for \( 1 \leq i \leq 4 \).
In order to prove Theorem 1, it is sufficient to show that

\[ \int_{S^{reg}} (E_i f(z))^p \leq C \int_{X \cap D} |f(\zeta)|^p dV_{n-1}. \]

Schmalz [SCH] obtained the following:

**Lemma 2** Let \( t(z, \zeta) = \text{Im} < w(z, \zeta), \zeta - z > \). We set \( \zeta_j = \xi_j + i\eta_j + i_n \) and \( E_\gamma(z) = \{ \zeta \in D \mid |\zeta - z| < \gamma \|d\rho(z)\| \} \) for all \( \gamma > 0 \). Then there are constants \( c > 0, \gamma > 0, \) and numbers \( \mu, \nu \in \{1, \ldots, 2n\} \) such that, \( \{\mu, \nu, \xi_1, \ldots, \eta_n, \xi_{2n}\} \) (\( \xi_\mu \) and \( \xi_\nu \) have to be omitted) forms a coordinate system in \( E_\gamma(z) \) \( \{\rho, t(z, \zeta), \eta_1, \ldots, \mu, \nu, \xi_1, \ldots, \eta_n\} \) forms a local coordinate system in \( E_\gamma(\zeta) \), respectively) and we have the estimates

\[ d\sigma(\zeta) \leq \frac{c}{\|d\rho(z)\|} |d\zeta_t(z, \zeta) \wedge \cdots \wedge d\zeta_{2n}| \quad \text{on} \quad S^{reg} \cap E_\gamma(z), \]

\[ d\sigma(z) \leq \frac{c}{\|d\rho(z)\|} |d\zeta_t(z, \zeta) \wedge \cdots \wedge d\eta_{2n}| \quad \text{on} \quad S^{reg} \cap E_\gamma(\zeta). \]

Using Lemma 1 and Lemma 2 we have the following:

**Lemma 3** Let \( 1 < p < \infty \) and \( f \in L^p(X \cap D) \cap C(X \cap D) \). Then there exists a constant \( C > 0 \) such that for \( \delta > 0 \) sufficiently small,

\[ \int_{S^{reg}} (E_i f(z))^p d\sigma(z) \leq C \int_{X \cap D} |f(\zeta)|^p dV_{n-1}(\zeta) \]

for \( i = 1, 2 \).

**Proof** In what follows we denote by \( C \) any positive constant which does not depend on the relevant parameters. By Hölder's inequality, we have

\[ \begin{align*}
E_i f(z) & \leq \left( \int_{X \cap D} |f(\zeta)|^p K_i(z, \zeta) dV_{n-1}(\zeta) \right)^{\frac{1}{p}} \left( \int_{X \cap D} K_i(z, \zeta) dV_{n-1}(\zeta) \right)^{\frac{1}{q}}.
\end{align*} \]

By Lemma 1 we have

\[ E_i f(z) \leq C \left( \int_{X \cap D} |f(\zeta)|^p K_i(z, \zeta) dV_{n-1}(\zeta) \right)^{\frac{1}{p}}. \]

Using Fubini's theorem, we have

\[ \int_{S^{reg}} (E_i f(z))^p d\sigma(z) \leq C \int_{X \cap D} |f(\zeta)|^p \left( \int_{S^{reg}} K_i(z, \zeta) d\sigma(z) \right) dV_{n-1}(\zeta). \]

Since \( \zeta \in X \), we have

\[ \int_{S^{reg}} K_1(z, \zeta) d\sigma(z) \leq C \int_{S^{reg}} \frac{|z_n|}{|\zeta - z|^{2n-1}} d\sigma(z) \leq C \int_{S^{reg}} \frac{1}{|\zeta - z|^{2n-2}} d\sigma(z) \leq C. \]
Moreover, we have

\[ \int_{S_{reg}} K_2(z, \zeta) d\sigma(z) \]
\[ \leq C \int_{S_{reg}} \frac{|z_n| \|d\rho(z)\|}{|\Phi||\Phi^*| |\zeta - z|^{2n-4}} d\sigma(z) \]
\[ \leq C \int_{z \in E_\nu(\zeta)} \frac{|z_n| \|d\rho(z)\|}{|\Phi||\Phi^*| |\zeta - z|^{2n-4}} d\sigma(z) + C \int_{z \notin E_\nu(\zeta)} \frac{|z_n| \|d\rho(z)\|}{|\Phi||\Phi^*| |\zeta - z|^{2n-4}} d\sigma(z) \]
\[ = I_1(\zeta) + I_2(\zeta) \]

By Lemma 2, we obtain

\[ I_1(\zeta) \leq C \int_{|t| < R} \frac{dt_1 \wedge \cdots \wedge dt_{2n-1}}{|t_1| + |t'|^2|t'|^{2n-5}} \]
\[ \leq C \int_{|t'| < R} \frac{dt_2 \wedge \cdots \wedge dt_{2n-1}}{|t'|^{2n-3}} \leq C, \]
\[ I_2(\zeta) \leq \int_{z \notin E_\nu(\zeta)} \frac{1}{|\zeta - z|^{2n-2}} d\sigma(z) \leq C. \]

Lemma 3 is proved.

In order to estimate integrals \( E_3f \) and \( E_4f \) we use the following lemma obtained by Henkin-Leiterer (see Lemma 3.2.4 [HER]). But we give a proof for the reader’s convenience.

**Lemma 4** There exist real valued quadratic polynomials \( P(z, \zeta) \) in the real coordinates of \( \zeta \), whose coefficients are \( C^1 \) functions in \( z \in \overline{U_2} \) such that the following estimates hold:

(i) \( P(z, \zeta) = \text{Im} F(z, \zeta) + o(|\zeta - z|^2) \) for \( \zeta, z \in V_2 \).

(ii) \( Q(z, \zeta) = \rho(\zeta) - \rho(z) + O(|\zeta - z|^3) \) for \( z, \zeta \in V_2 \).

(iii) \( \|d_c P(z, \zeta) \wedge d_c Q(z, \zeta)\| \geq \frac{1}{\sqrt{n}} \|d\rho(z)\|^2 - C(\|d\rho(z)\| |\zeta - z| + |\zeta - z|^2) \) for \( z, \zeta \in V_2 \).

(iv) \( |\Phi(z, \zeta)| \geq C(|P(z, \zeta)| + |Q(z, \zeta)| + |\zeta - z|^2) \) for \( z \in V_2 \cap \overline{D}, \zeta \in \partial D \).

(v) \( |\Phi(z, \zeta)| \geq C(|P(z, \zeta)| + |Q(z, \zeta)| + |\zeta - z|^2) \) for \( z, \zeta \in V_2 \cap \overline{D} \).

(vi) \( |P(z, \zeta)| + |\zeta - z|^2 \approx |P(z, \zeta)| + |\zeta - z|^2 \) for \( \zeta, z \in \overline{D} \cap V_2 \).

(vii) \( |Q(z, \zeta)| + |\zeta - z|^2 \approx |Q(z, \zeta)| + |\zeta - z|^2 \) for \( \zeta \in \overline{D} \cap V_2, z \in \partial D \).

**Proof** Let \( z_j = x_j + ix_{n+j}, \zeta_j = \xi_j + i\xi_{n+j} \). Since

\[ F(z, \zeta) = \sum_{j=1}^{n} \frac{\partial \rho}{\partial \zeta_j}(\zeta_j - z_j) - \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial \zeta_j \partial \zeta_k}(\zeta_j - z_j)(\zeta_k - z_k), \]
we obtain
\[ \text{Im } F(z, \zeta) = \sum_{j=1}^{n} \left\{ \frac{\partial \rho}{\partial x_j}(\zeta) (\xi_{j+n} - x_{j+n}) - \frac{\partial \rho}{\partial x_j}(\xi_j - x_j) \right\} \\
+ \sum_{j,k=1}^{2n} u_{j,k}(\zeta) (\xi_j - x_j)(\xi_k - x_k), \]
where \( u_{j,k} \) are \( C^1 \) functions in \( \nu_2 \). We set
\[ P(z, \zeta) = \sum_{j=1}^{n} \left\{ \frac{\partial \rho}{\partial x_j}(z) + \sum_{s=1}^{2n} \frac{\partial^2 \rho}{\partial x_j \partial x_s}(z) (\xi_s - x_s) \right\} (\xi_{j+n} - x_{j+n}) \\
- \left\{ \frac{\partial \rho}{\partial x_{j+n}}(z) + \sum_{s=1}^{2n} \frac{\partial^2 \rho}{\partial x_{j+n} \partial x_s}(z) (\xi_s - x_s) \right\} (\xi_j - x_j) \\
+ \sum_{j,k=1}^{2n} u_{j,k}(z) (\xi_j - x_j)(\xi_k - x_k). \]

Then
\[ \text{Im } F(z, \zeta) - P(z, \zeta) \\
= \sum_{j=1}^{n} \left\{ \frac{\partial \rho}{\partial x_j}(\zeta) - \frac{\partial \rho}{\partial x_j}(z) - \sum_{s=1}^{2n} \frac{\partial^2 \rho}{\partial x_j \partial x_s}(z) (\xi_s - x_s) \right\} (\xi_{j+n} - x_{j+n}) \\
- \sum_{j=1}^{n} \left\{ \frac{\partial \rho}{\partial x_{j+n}}(\zeta) - \frac{\partial \rho}{\partial x_{j+n}}(z) - \sum_{s=1}^{2n} \frac{\partial^2 \rho}{\partial x_{j+n} \partial x_s}(z) (\xi_s - x_s) \right\} (\xi_j - x_j) \\
+ o(|\zeta - z|^2). \]
This proves (i). We set
\[ Q(z, \zeta) = \sum_{j=1}^{2n} \frac{\partial \rho}{\partial x_j}(z) (\xi_j - x_j) + \frac{1}{2} \sum_{j,k=1}^{2n} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(z) (\xi_j - x_j)(\xi_k - x_k). \]
It follows from Taylor's formula that
\[ \rho(\zeta) - \rho(z) = Q(z, \zeta) + O(|\zeta - z|^3). \]
This proves (ii). Since
\[ d_\zeta P(z, \zeta) \wedge d_\zeta Q(z, \zeta) \\
= \sum_{j,k=1}^{2n} \left\{ \left( \frac{\partial \rho}{\partial x_{j+n}}(\zeta) d\xi_j + \frac{\partial \rho}{\partial x_j}(\zeta) d\xi_{j+n} + O(|\zeta - z|) \right) \right\} \\
\times \left\{ \left( \frac{\partial \rho}{\partial x_k}(\zeta) d\xi_k + \frac{\partial \rho}{\partial x_{k+n}}(\zeta) d\xi_{k+n} + O(|\zeta - z|) \right) \right\} \\
= \sum_{j=1}^{n} \left\{ \left( \frac{\partial \rho}{\partial x_j}(\zeta) \right)^2 + \left( \frac{\partial \rho}{\partial x_{j+n}} \right)^2 \right\} d\xi_{j+n} \wedge d\xi_j + \cdots, \]
we obtain
\[ \|d_\zeta P(z,\zeta) \wedge d_\zeta Q(z,\zeta)\| \geq \frac{1}{\sqrt{n}} \|d\rho(\zeta)\|^2 - C(\|d\rho(\zeta)\| |\zeta - z| + |\zeta - z|^2). \]

This proves (iii). In view of Proposition 1 (i) and (iii), we have for \( z \in V_2 \cap \overline{D} \) and \( \zeta \in \partial D \),
\[
|\Phi(z,\zeta)| \geq C|F(z,\zeta)| \geq C(|\text{Im} F(z,\zeta)| + |\text{Re} F(z,\zeta)|) \geq C(|P(z,\zeta)| + |Q(z,\zeta)| + |\zeta - z|^2).
\]
This proves (iv). Similarly, we can prove (v), (vi) and (vii). Lemma 4 is proved.

**Definition 5** For \( \xi \in \partial D \) and \( \delta > 0 \), define
\[
T_\xi := \{ \zeta \in \mathbb{C}^n | \sum_{j=1}^{n} \frac{\partial\rho(\xi)}{\partial \xi_j} (\zeta_j - \xi_j) = 0 \},
\]
\[
B(\xi,\delta) := \{ \zeta \in \mathbb{C}^n | |\zeta - \xi| < \delta \},
\]
\[
\tilde{H}_\xi(\delta) := B(\xi,\delta) \cap \{ \zeta \in \mathbb{C}^n | |d\rho(\xi)||\text{dist}(\zeta,T_\xi) < \delta^2 \},
\]
\[
H_\xi(\delta) := \tilde{H}_\xi(\delta) \cap \overline{D}.
\]

\( H_\xi(\delta) \) is called the Hörmander ball of radius \( \delta \) with center \( \xi \).

Then Henkin-Leiterer (see Lemma 3.6.5 [HER]) proved the following:

**Lemma 5** There exists a number \( \delta > 0 \) with the following properties:
\[
\|d_{\zeta'}\rho(z)\| |\zeta' - z'| \geq \left| \frac{\partial\rho}{\partial z_n}(z)z_n \right|
\]
\[
\|d_{\zeta'} P(z,\zeta) \wedge d_{\zeta'} Q(z,\zeta)\| \geq \frac{1}{\sqrt{2n}} \|d_{\zeta'}\rho(z)\|^2
\]
for all \( z \in \partial D \setminus X \) and \( \zeta \in H_x \left( \delta \left| \frac{\partial\rho}{\partial z_n}(z)z_n \right|^{1/2} \right) \cap V_2 \cap X \).

Now we shall prove the following:

**Lemma 6** For \( z \in \partial \Omega \setminus X \) and any positive number \( \varepsilon \) with \( 0 < \varepsilon < 1/2 \), we have
\[
\int_{\Omega \cap D} |K_1(z,\zeta)||Q(z,\zeta)|^{-\varepsilon} dV_{n-1}(\zeta) \leq C \varepsilon |z_n|^{-2\varepsilon}
\]
for \( i = 3,4 \).

**Proof** Using the method of Henkin-Leiterer (Lemma 3.6.6 [HER]), we have
\[
\int_{\Omega \cap D} K_3(z,\zeta)|Q(z,\zeta)|^{-\varepsilon} dV_{n-1}(\zeta)
\]
\[
\leq C \int_{|t| < R} \frac{|z_n| |Q(z,\zeta)|^{-\varepsilon} |d_{\zeta'} P(z,\zeta) \wedge d_{\zeta'} Q(z,\zeta)|}{(|P(z,\zeta)| + |Q(z,\zeta)| + |\zeta - z|^2)^3 |\zeta - z|^{2n-5} dV_{n-1}(\zeta)}
\]
\[
\leq C \int_{|t| < R} \frac{|z_n| |t_1|^{-\varepsilon}}{(|z_n|^2 + |t_1|^2 + |t_2|^2)^{2n-5} dt_1 \cdots dt_2 dV_{n-2}}
\]
where $t = (t_1, \cdots, t_{2n-2})$. We set $t' = (t_3, \cdots, t_{2n-2})$. Then we obtain for some $R > 0$,

$$
\int_{\zeta \in \chi \cap D} K_3(z, \zeta)|Q(z, \zeta)|^{-\epsilon} dV_{n-1}(\zeta)
\leq C \int_{|t|<R} \frac{|z_n||t_1|^{-\epsilon}}{(|z_n|^2 + |t_1| + |t_2| + |t'|^2)^3|t'|^2n-5} dt_1 \cdots dt_{2n-2} 
\leq C \int_0^R \int_0^R \int_0^R \frac{|z_n|t_1^{-\epsilon}}{(|z_n|^2 + t_1 + t_2 + r^2)^3} dt_1 dt_2 dr 
\leq C \int_0^R \int_0^R \frac{|z_n|t_1^{-\epsilon}}{(|z_n|^2 + t_1 + r^2)^2} dt_1 dr 
\leq C |z_n|^{-2\epsilon} \int_0^\infty \int_0^\infty \frac{u^{1-2\epsilon}}{(1 + u^2 + v^2)^2} dudv 
\leq C\epsilon |z_n|^{-2\epsilon}.
$$

We write

$$
H_z := H_z \left( \delta \left| \frac{\partial \rho(z)}{\partial z_n}(z) z_n \right|^{\frac{1}{2}} \right).
$$

and

$$
\alpha = \left| \frac{\partial \rho(z)}{\partial z_n}(z) \right| z_n.
$$

Then we have

$$
\int_{\zeta \in (\chi \cap D) \setminus H_z} |Q(z, \zeta)|^{-\epsilon} K_4(z, \zeta) dV_{n-1}(\zeta)
\leq \int_{\zeta \in (\chi \cap D) \setminus H_z} \frac{\alpha |Q(z, \zeta)|^{-\epsilon} \|d^\zeta Q(z, \zeta)\|}{(\alpha + |Q(z, \zeta)| + |z - z'|^2)^3|z - z'|^{2n-5} dV_{n-1}(\zeta)} 
\leq C \int_{|t|<R} \frac{\alpha |t_1|^{-\epsilon}}{(\alpha + |z_n|^2 + |t_1| + |t'|^2)^3|t'|^2n-5} dt_1 \cdots dt_{2n-2} 
\leq C \int_0^R \frac{\alpha t_1^{-\epsilon}}{(\alpha + |z_n|^2 + t_1)^2} dt_1 
\leq C |z_n|^{-2\epsilon} \int_0^\infty \frac{x^{-\epsilon}}{(1 + x)^2} dx \leq C\epsilon |z_n|^{-2\epsilon}.
$$

On the other hand we set

$$
J(z) = \int_{\zeta \in H_z \cap (\chi \cap D)} |Q(z, \zeta)|^{-\epsilon} K_4(z, \zeta) dV_{n-1}(\zeta)
$$

and

$$
\beta = \frac{\alpha}{\|d^\zeta \rho(z)\|}.
$$
Then we obtain
\[
\|d_{z'}\rho(z)\|J(z)
\leq C \int_{\xi \in H_{\mathbb{R}} \cap (\mathbb{C} \times D)} \left( \beta^2 + |\xi|^2 + |P| + |Q| + |\zeta - z|^2 \right)^3 |\zeta - z|^{2n-5} dV_{n-1}(\zeta)
\]

We set \( b = \sqrt{\beta^2 + |z_n|^2} \). Then we have
\[
\|d_{z'}\rho(z)\|J(z) \leq C \int_{|t| < R} \frac{\alpha t_1^{-\epsilon}}{(b^2 + t_1 + t_2 + |t'|^2)^3 |t'|^{2n-5}} dt_1 \cdots dt_{2n-2}
\]
\[
\leq C \int_0^R dt_1 \int_0^R \frac{\alpha t_1^{-\epsilon}}{(b^2 + t_1 + r^2)^2} dr
\]
\[
\leq C \int_0^R dy \int_0^R \frac{\alpha y^{-2\epsilon}}{(b^2 + y^2 + r^2)^2} dr
\]
\[
\leq C \int_0^{\infty} \frac{\alpha b^{-2\epsilon} x^{-2\epsilon}}{(1 + x^2)^2} dx
\]
\[
\leq C_\epsilon |z_n|^{-2\epsilon} \|d_{z'}\rho(z)\|.
\]

Lemma 6 is proved.

**Lemma 7** For \( \zeta \in X \cap D \), \( 0 < \epsilon < 1/2 \) and \( i = 3, 4 \), there exists a positive constant \( C_\epsilon \) which depends only on \( \epsilon \) such that
\[
\int_{S^{n-1}} |K_i(z, \zeta)||z_n|^{-2\epsilon} d\sigma(z) \leq C_\epsilon |\rho(\zeta)|^{-\epsilon}.
\]

**Proof** We set
\[
K_5(z, \zeta) = \frac{\|d\rho(z)\|^2 |z_n|}{|\Phi(x, \zeta)|^2 |\Phi^*(x, \zeta)| |\zeta - z|^{2n-5}}.
\]
Since \( \|d_{z'}\rho(z)\| \leq \|d\rho(z)\| \) and \( \frac{\partial \rho}{\partial z_n}(z) \leq \|d\rho(z)\| \), it is sufficient to show that
\[
\int_{S^{n-1}} |K_5(z, \zeta)||z_n|^{-2\epsilon} d\sigma(z) \leq C_\epsilon |\rho(\zeta)|^{-\epsilon}.
\]

We set
\[
L_1(\zeta) = \int_{z \in S^{n-1} \cap E_\zeta(\zeta)} |K_5(z, \zeta)||z_n|^{-2\epsilon} d\sigma(z)
\]
and
\[
L_2(\zeta) = \int_{z \in S^{n-1} \setminus E_\zeta(\zeta)} |K_5(z, \zeta)||z_n|^{-2\epsilon} d\sigma(z).
\]
Then we obtain by Lemma 2,
Similarly, we have \( L_2(\zeta) \leq C_\varepsilon |\rho(\zeta)|^{-\varepsilon} \), which completes the proof of Lemma 7.

Using the same technique as in the proof in Adachi [AD2], we obtain the following lemma. We omit the proof.

**Lemma 8** Let \( D \) be a strictly pseudoconvex domain in \( \mathbb{C}^n \) (with not necessarily smooth boundary). Let \( f \) be an \( L^p \) (\( 1 \leq p < \infty \)) holomorphic function in \( D \) and let \( \varphi \) be a \( C^\infty \) function in \( \mathbb{C}^n \). Then

\[
L_D(\varphi f)(z) = \frac{n!}{(2\pi i)^n} \int_D f(\zeta)\varphi(\zeta) \wedge \left( \frac{\chi(\zeta)w_j(z,\zeta)}{\Phi(z,\zeta)} \right) \wedge \omega(\zeta)
\]

is an \( L^p \) holomorphic function in \( D \).

### 3 Proof of Theorem 1

By Lemma 8 and the proof of Theorem 4.11.1 in Henkin-Leiterer [HER], we may assume that \( X = \{ z \in \mathbb{C}^n \mid z_n = 0 \} \). Let \( q \) be a positive number such that \( 1/p + 1/q = 1 \). We choose \( \varepsilon > 0 \) such that \( \max\{\varepsilon p, \varepsilon q\} < 1/2 \). From now on we denote by \( C_\varepsilon \) any positive constant which depends only on \( \varepsilon \). It is sufficient to show that

\[
\int_{S^{n-1}} |E_i f(z)|^p d\sigma(z) \leq C_\varepsilon \int_{X \cap D} |f(\zeta)|^p dV_{n-1}(\zeta).
\]

for \( i = 3, 4 \). By Lemma 6 and Hölder’s inequality, we obtain for \( i = 3, 4 \),

\[
|E_i f(z)| \leq \int_{X \cap D} |f(\zeta)||K_i(z,\zeta)||Q(z,\zeta)|^p|Q(z,\zeta)|^{-\varepsilon} dV_{n-1}(\zeta)
\leq \left( \int_{X \cap D} |f(\zeta)|^p|K_i(z,\zeta)||Q(z,\zeta)|^{\varepsilon p} dV_{n-1}(\zeta) \right)^{1/p} \times \left( \int_{X \cap D} |K_i(z,\zeta)||Q(z,\zeta)|^{-\varepsilon q} dV_{n-1}(\zeta) \right)^{1/q}
\leq C_\varepsilon |z_n|^{-2\varepsilon} \left( \int_{X \cap D} |f(\zeta)|^p|K_i(z,\zeta)||Q(z,\zeta)|^{\varepsilon p} dV_{n-1}(\zeta) \right)^{1/2}.
\]
Consequently,

$$|E_i(z)|^p \leq C_\varepsilon |z_n|^{-2\varepsilon p} \left( \int_{X \cap D} |f(\zeta)^p|K(z, \zeta)||Q(z, \zeta)|^\varepsilon dV_{n-1}(\zeta) \right).$$

Using Fubini’s theorem, Lemma 4(ii) and Lemma 7, we have

$$\int_{S_{reg}} |Ef(z)|^p d\sigma(z) \leq C_\varepsilon \int_{X \cap D} |f(\zeta)|^p \left\{ \int_{S_{reg}} |z_n|^{-2\varepsilon p} |K_i(z, \zeta)||Q(z, \zeta)|^\varepsilon d\sigma(z) \right\} dV_{n-1}(\zeta)$$

$$\leq C_\varepsilon \int_{X \cap D} |f(\zeta)|^p \left\{ \int_{S_{reg}} |z_n|^{-2\varepsilon p} |K_i(z, \zeta)||\rho(\zeta)|^\varepsilon d\sigma(z) \right\} dV_{n-1}(\zeta)$$

$$+ C_\varepsilon \int_{X \cap D} |f(\zeta)|^p \left\{ \int_{S_{reg}} |z_n|^{-2\varepsilon p} |K_i(z, \zeta)||z - \zeta|^\varepsilon d\sigma(z) \right\} dV_{n-1}(\zeta)$$

$$\leq C_\varepsilon \int_{X \cap D} |f(\zeta)|^p dV_{n-1}(\zeta)$$

$$+ C_\varepsilon \int_{X \cap D} |f(\zeta)|^p \left\{ \int_{S_{reg}} |z_n|^{-2\varepsilon p} |K_i(z, \zeta)||z - \zeta|^3\varepsilon d\sigma(z) \right\} dV_{n-1}(\zeta).$$

We set

$$T_i(\zeta) = \int_{S_{reg}} |z_n|^{-2\varepsilon p} |K_i(z, \zeta)||z - \zeta|^3\varepsilon d\sigma(z).$$

In order to prove the inequality $|T_i(\zeta)| \leq C_\varepsilon$, it is sufficient to show that

$$T(\zeta) := \int_{S_{reg}} \frac{|z_n|^{1-2\varepsilon p} ||d\rho(z)||^2 |\zeta - z|^{3\varepsilon}}{|\Phi|^2 |\Phi^*||\zeta - z|^{2n-5}} d\sigma(z) \leq C_\varepsilon.$$  

Then we have

$$I_1(\zeta) = \int_{z \in E_\gamma(\zeta) \cap S_{reg}} \frac{|z_n|^{1-2\varepsilon p} ||d\rho(z)||^2 |\zeta - z|^{3\varepsilon}}{|\Phi|^2 |\Phi^*||\zeta - z|^{2n-5}} d\sigma(z)$$

$$+ \int_{z \notin E_\gamma(\zeta) \cap S_{reg}} \frac{|z_n|^{1-2\varepsilon p} ||d\rho(z)||^2 |\zeta - z|^{3\varepsilon}}{|\Phi|^2 |\Phi^*||\zeta - z|^{2n-5}} d\sigma(z)$$

$$= I_{11}(\zeta) + I_{12}(\zeta).$$

In view of Lemma 2, we have by setting $t' = (t_2, \cdots, t_{2n-1})$

$$I_{11}(\zeta) \leq C \int_{|t| < R} \frac{dt_1 \cdots dt_{2n-1}}{(|\rho(\zeta)| + |t_1| + |t'|^{5/2})^{\varepsilon p + (5/2)}}|t'|^{2n-5-3\varepsilon}.$$ 

Using the polar coordinate change, we obtain

$$I_{11}(\zeta) \leq C \int_0^R \frac{r^{2+3\varepsilon}}{(|\rho(\zeta)| + r^{5/2})^{\varepsilon p + (5/2)}} dr.$$
We set \( \sqrt{\rho(z)}|y = r \). Then we obtain

\[
I_{11}(\zeta) \leq C|\rho(\zeta)|^{ep/2} \int_0^r \frac{y^{2+3ep}}{(1+y^2)^{ep+(3/2)}} dy \leq C_{\varepsilon}.
\]

Similarly, we obtain

\[
I_{12}(z) \leq \int_{z \in E_n(\zeta) \cap S^{n+q}} \frac{|z_n|^{1-2ep}||z - \zeta|^{2+3ep}}{||\Phi||^2 ||\Phi^*|| ||z - \zeta|^{2n-5}} d\sigma(z)
\]

\[
\leq \int_{z \in E_n(\zeta) \cap S^{n+q}} \frac{d\sigma(z)}{||z - \zeta||^{2n-2-ep}} \leq C_{\varepsilon}.
\]

Therefore, Theorem 1 is proved.

**Remark 2** If \( D \) is a strictly pseudoconvex domain with \( C^\infty \) boundary and if \( X \) intersects \( \partial D \) transversally, Adachi [AD1] and Elgueta [ELG] proved that for any holomorphic function \( f \) in \( X \cap D \) that is of class \( C^\infty \) on \( X \cap D \) there exists a holomorphic function \( g \) in \( D \) that is of class \( C^\infty \) on \( D \) such that \( f = g \) on \( X \cap D \). In case \( D \) is a strictly pseudoconvex domain with non-smooth boundary, the \( C^\infty \) extension problem is still open.

**References**